

SPINORIAL REPRESENTATION OF SUBMANIFOLDS IN METRIC LIE GROUPS

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ABSTRACT. In this paper we give a spinorial representation of submanifolds of any dimension and codimension into Lie groups equipped with left invariant metrics. As applications, we get a spinorial proof of the Fundamental Theorem for submanifolds into Lie groups, we recover previously known representations of submanifolds in \mathbb{R}^n and in the 3-dimensional Lie groups S^3 and $E(\kappa, \tau)$, and we get a new spinorial representation for surfaces in the 3-dimensional semi-direct products: this achieves the spinorial representations of surfaces in the 3-dimensional homogeneous spaces. We finally indicate how to recover a Weierstrass-type representation for CMC-surfaces in 3-dimensional metric Lie groups recently given by Meeks, Mira, Perez and Ros.

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1. INTRODUCTION

The purpose of this paper is to give a spinorial representation of an isometric immersion of a Riemannian manifold M into a Lie group G equipped with a left invariant metric. The result is roughly the following: if M is a simply connected Riemannian manifold, E is a real vector bundle on M equipped with a fibre metric and a compatible connection, and $B : TM \times TM \rightarrow E$ is bilinear and symmetric, then an isometric immersion of M into G with normal bundle E and second fundamental form B is equivalent to a spinor field φ solution of a Killing-type equation on M ; the spinor bundle is constructed from the Clifford algebra of the metric Lie algebra \mathcal{G} of the group, and the immersion is explicitly obtained by the integration of a \mathcal{G} -valued 1-form on M defined in terms of the spinor field φ . A precise statement with the suitable necessary hypotheses is given in Section 3 of the paper.

The explicit representation formula of the immersion in terms of the spinor field may be considered as a generalized Weierstrass representation formula for manifolds into metric Lie groups.

We then give some applications of this result. We first obtain an easy proof of a theorem by Piccione and Tausk [18]: under suitable hypotheses, the necessary equations of Gauss, Codazzi and Ricci are also sufficient to obtain an immersion of a simply connected manifold into a metric Lie group. We then show how our general result permits to recover the known spinorial representation for submanifolds in \mathbb{R}^n [4], and also obtain a new spinorial representation for submanifolds in \mathbb{H}^n considered as a metric Lie group. We finally study more precisely the case of surfaces in a 3-dimensional metric Lie group: we recover the known spinorial representations in S^3 [16] and $E(\kappa, \tau)$ [19], and obtain a new spinorial representation of

surfaces in a general semi-direct product; this especially includes the cases of surfaces into the groups Sol_3 and $\mathbb{H}^2 \times \mathbb{R}$, which achieves the spinorial representations of surfaces into the 3-dimensional homogeneous spaces initiated in [8, 16, 19]. We also deduce alternative proofs of the Fundamental Theorems for surfaces in $E(\kappa, \tau)$ by Daniel [7] and in Sol_3 by Lodovici [11]. We finish the paper showing how the general spinorial representation formula permits to recover the recent Weierstrass-type representation formula by Meeks, Mira, Perez and Ros [13, Theorem 3.12] concerning constant mean curvature surfaces in 3-dimensional metric Lie groups.

The main result of the paper thus gives a general framework for a variety of Weierstrass-type representation formulas existing in the literature, and is also a tool to get representation formulas in new contexts.

We quote the following related papers: Friedrich obtained in [8] a geometric spinorial representation of a surface in \mathbb{R}^3 showing that a surface in \mathbb{R}^3 may be represented by a constant spinor field of \mathbb{R}^3 restricted to the surface; this result was then extended to S^3 and \mathbb{H}^3 by Morel [16] and to other 3-dimensional homogeneous spaces by Roth [19]. It was then extended by Bayard, Lawn and Roth to surfaces in dimension 4 [3] and afterwards to manifolds in \mathbb{R}^n [4]. Spinorial representation were also studied in pseudo-Riemannian spaces, by Lawn in $\mathbb{R}^{2,1}$ [9], Lawn and Roth in 3-dimensional Lorentzian space forms [10], Bayard in $\mathbb{R}^{3,1}$ [1], Bayard and Patty [5] and Patty [17] in $\mathbb{R}^{2,2}$. Close to the purpose of the paper, Berdinskii and Taimanov gave in [6] a spinorial representation for a surface in a 3-dimensional metric Lie group.

The outline of the paper is as follows: Section 2 is dedicated to preliminaries concerning notation and spin geometry of a submanifold in a metric Lie group, Section 3 to the statement and the proof of the main theorem, and Section 4 to a spinorial proof of the Fundamental Theorem for submanifolds in a metric Lie group. We then give further applications in Section 5: we study the cases of a submanifold in \mathbb{R}^n and \mathbb{H}^n , and of a hypersurface in a general metric Lie group, specifying further to the cases of a surface in S^3 , $E(\kappa, \tau)$ and a semi-direct product, as Sol_3 and $\mathbb{H}^2 \times \mathbb{R}$. We finally consider the case of a CMC-surface in a 3-dimensional metric Lie group. An appendix ends the paper concerning the links between the Clifford product and some natural operations on skew-symmetric operators.

2. PRELIMINARIES

2.1. Notations. Let G be a Lie group, endowed with a left invariant metric $\langle \cdot, \cdot \rangle$, and \mathcal{G} its Lie algebra: \mathcal{G} is the space of the left invariant vector fields on G , equipped with the Lie bracket $[\cdot, \cdot]$ and is identified to the linear space tangent to G at the identity. We consider the Maurer-Cartan form $\omega_G \in \Omega^1(G, \mathcal{G})$ defined by

$$(1) \quad \omega_G(v) = L_{g^{-1}*}(v) \quad \in \mathcal{G}$$

for all $v \in T_g G$, where $L_{g^{-1}}$ denotes the left multiplication by g^{-1} on G and $L_{g^{-1}*} : T_g G \rightarrow \mathcal{G}$ is its differential. This form induces a bundle isomorphism

$$(2) \quad \begin{aligned} TG &\rightarrow G \times \mathcal{G} \\ (g, v) &\mapsto (g, \omega_G(v)). \end{aligned}$$

which preserves the fibre metrics. We note that a vector field $X \in \Gamma(TG)$ is left invariant if, by (2), $X : G \rightarrow \mathcal{G}$ is a constant map. Let us consider the Levi-Civita

connection ∇^G of $(G, \langle \cdot, \cdot \rangle)$ and the linear map

$$\begin{aligned} \Gamma : \quad \mathcal{G} &\rightarrow \Lambda^2 \mathcal{G} \\ X &\mapsto \Gamma(X) \end{aligned}$$

such that, for all $X, Y \in \mathcal{G}$

$$(3) \quad \nabla_X^G Y = \Gamma(X)(Y).$$

By the Koszul formula, Γ is determined by the metric as follows: for all $X, Y, Z \in \mathcal{G}$,

$$(4) \quad \langle \Gamma(X)(Y), Z \rangle = \frac{1}{2} \langle [X, Y], Z \rangle + \frac{1}{2} \langle [Z, X], Y \rangle - \frac{1}{2} \langle [Y, Z], X \rangle.$$

Since ∇^G is without torsion, we have, for all $X, Y \in \mathcal{G}$,

$$(5) \quad \Gamma(X)(Y) - \Gamma(Y)(X) = [X, Y].$$

We note that the curvature of ∇^G is given by

$$(6) \quad R^G(X, Y) = [\Gamma(X), \Gamma(Y)] - \Gamma([X, Y]) \in \Lambda^2 \mathcal{G}$$

for all $X, Y \in \mathcal{G}$. In the formula the first brackets stand for the commutator of the endomorphisms.

2.2. The spinor bundle of G . Let us denote by $Cl(\mathcal{G})$ the Clifford algebra of \mathcal{G} with its scalar product, and let us consider the representation

$$\begin{aligned} \rho : \quad Spin(\mathcal{G}) &\rightarrow GL(Cl(\mathcal{G})) \\ a &\mapsto \xi \mapsto a\xi. \end{aligned}$$

This representation is a real representation and is not irreducible in general: it is a sum of irreducible representations [12]. By (2) the principal bundle Q_G of the positively oriented and orthonormal frames of G is trivial

$$Q_G \simeq G \times SO(\mathcal{G}),$$

and we may consider the trivial spin structure

$$\tilde{Q}_G := G \times Spin(\mathcal{G})$$

and the corresponding spinor bundle

$$\Sigma := \tilde{Q}_G \times_{\rho} Cl(\mathcal{G}) \simeq G \times Cl(\mathcal{G}).$$

We will say that a spinor field $\varphi \in \Gamma(\Sigma)$ is *left invariant* if it is constant as a map $G \rightarrow Cl(\mathcal{G})$. The covariant derivative of a left invariant spinor field is

$$(7) \quad \nabla_X^G \varphi = \frac{1}{2} \Gamma(X) \cdot \varphi$$

where $\Gamma(X) \in \Lambda^2 \mathcal{G} \subset Cl(\mathcal{G})$ and the dot " \cdot " stands for the Clifford product.

2.3. The spin representation of $Spin(p) \times Spin(q)$. Let us assume that $p+q = n$, and fix an orthonormal basis $e_1^o, e_2^o, \dots, e_n^o$ of \mathcal{G} ; this gives a splitting $\mathcal{G} = \mathbb{R}^p \oplus \mathbb{R}^q$ (the first factor corresponds to the first p vectors, and the second factor to the last q vectors of the basis) and a natural map

$$Spin(p) \times Spin(q) \rightarrow Spin(\mathcal{G}), \quad (a_p, a_q) \mapsto a := a_p \cdot a_q$$

associated to the isomorphism

$$Cl(\mathcal{G}) = Cl_p \hat{\otimes} Cl_q.$$

We thus also have a representation, still denoted by ρ ,

$$(8) \quad \begin{aligned} \rho : \quad Spin(p) \times Spin(q) &\rightarrow GL(Cl(\mathcal{G})) \\ (a_p, a_q) &\mapsto \xi \mapsto a\xi. \end{aligned}$$

2.4. The twisted spinor bundle. We consider a p -dimensional Riemannian manifold M and a bundle $E \rightarrow M$ of rank q , with a fibre metric and a compatible connection. We assume that E and TM are oriented and spin, with given spin structures

$$\tilde{Q}_M \xrightarrow{2:1} Q_M \quad \text{and} \quad \tilde{Q}_E \xrightarrow{2:1} Q_E$$

where Q_M and Q_E are the bundles of positively oriented orthonormal frames of TM and E , and we set

$$\tilde{Q} := \tilde{Q}_M \times_M \tilde{Q}_E;$$

this is a $Spin(p) \times Spin(q)$ principal bundle. We define

$$\Sigma := \tilde{Q} \times_{\rho} Cl(\mathcal{G})$$

and

$$U\Sigma := \tilde{Q} \times_{\rho} Spin(\mathcal{G}) \subset \Sigma$$

where ρ is the representation (8). Similarly to the usual construction in spin geometry, if we consider the representation

$$Ad : \quad Spin(p) \times Spin(q) \rightarrow Spin(\mathcal{G}) \xrightarrow{2:1} SO(\mathcal{G}) \rightarrow GL(Cl(\mathcal{G}))$$

and the Clifford bundle

$$Cl(TM \oplus E) = \tilde{Q} \times_{Ad} Cl(\mathcal{G}),$$

there is a Clifford action of $Cl(TM \oplus E)$ on Σ ; this action will be denoted below by a dot " \cdot ". The vector bundle Σ is moreover equipped with the covariant derivative ∇ naturally associated to the spinorial connections on \tilde{Q}_M and \tilde{Q}_E . Let us denote by $\tau : Cl(\mathcal{G}) \rightarrow Cl(\mathcal{G})$ the anti-automorphism of $Cl(\mathcal{G})$ such that

$$\tau(x_1 \cdot x_2 \cdots x_k) = x_k \cdots x_2 \cdot x_1$$

for all $x_1, x_2, \dots, x_k \in \mathcal{G}$, and set

$$(9) \quad \begin{aligned} \langle\langle \cdot, \cdot \rangle\rangle : \quad Cl(\mathcal{G}) \times Cl(\mathcal{G}) &\rightarrow Cl(\mathcal{G}) \\ (\xi, \xi') &\mapsto \tau(\xi')\xi. \end{aligned}$$

This map is $Spin(\mathcal{G})$ -invariant: for all $\xi, \xi' \in Cl(\mathcal{G})$ and $g \in Spin(\mathcal{G})$ we have

$$\langle\langle g\xi, g\xi' \rangle\rangle = \tau(g\xi')g\xi = \tau(\xi')\tau(g)g\xi = \tau(\xi')\xi = \langle\langle \xi, \xi' \rangle\rangle,$$

since $Spin(\mathcal{G}) \subset \{g \in Cl^0(\mathcal{G}) : \tau(g)g = 1\}$; this map thus induces a $Cl(\mathcal{G})$ -valued map

$$(10) \quad \begin{aligned} \langle\langle \cdot, \cdot \rangle\rangle : \quad \Sigma \times \Sigma &\rightarrow Cl(\mathcal{G}) \\ (\varphi, \varphi') &\mapsto \langle\langle [\varphi], [\varphi'] \rangle\rangle \end{aligned}$$

where $[\varphi]$ and $[\varphi'] \in Cl(\mathcal{G})$ represent φ and φ' in some spinorial frame $\tilde{s} \in \tilde{Q}$.

Lemma 2.1. *The map $\langle\langle \cdot, \cdot \rangle\rangle : \Sigma \times \Sigma \rightarrow Cl(\mathcal{G})$ satisfies the following properties: for all $\varphi, \psi \in \Gamma(\Sigma)$ and $X \in \Gamma(TM \oplus E)$,*

$$(11) \quad \langle\langle \varphi, \psi \rangle\rangle = \tau \langle\langle \psi, \varphi \rangle\rangle$$

and

$$(12) \quad \langle\langle X \cdot \varphi, \psi \rangle\rangle = \langle\langle \varphi, X \cdot \psi \rangle\rangle.$$

Proof. We have

$$\langle\langle \varphi, \psi \rangle\rangle = \tau[\psi] [\varphi] = \tau(\tau[\varphi] [\psi]) = \tau \langle\langle \psi, \varphi \rangle\rangle$$

and

$$\langle\langle X \cdot \varphi, \psi \rangle\rangle = \tau[\psi] [X][\varphi] = \tau([X][\psi])[\varphi] = \langle\langle \varphi, X \cdot \psi \rangle\rangle$$

where $[\varphi]$, $[\psi]$ and $[X] \in Cl(\mathcal{G})$ represent φ , ψ and X in some given frame $\tilde{s} \in \tilde{Q}$. \square

Lemma 2.2. *The connection ∇ is compatible with the product $\langle\langle \cdot, \cdot \rangle\rangle$:*

$$\partial_X \langle\langle \varphi, \varphi' \rangle\rangle = \langle\langle \nabla_X \varphi, \varphi' \rangle\rangle + \langle\langle \varphi, \nabla_X \varphi' \rangle\rangle$$

for all $\varphi, \varphi' \in \Gamma(\Sigma)$ and $X \in \Gamma(TM)$.

Proof. If $\varphi = [\tilde{s}, [\varphi]]$ is a section of $\Sigma = \tilde{Q} \times_\rho Cl(\mathcal{G})$, we have

$$(13) \quad \nabla_X \varphi = [\tilde{s}, \partial_X [\varphi] + \rho_*(\tilde{s}^* \alpha(X))([\varphi])], \quad \forall X \in TM,$$

where ρ is the representation (8) and α is the connection form on \tilde{Q} ; the term $\rho_*(\tilde{s}^* \alpha(X))$ is an endomorphism of $Cl(\mathcal{G})$ given by the multiplication on the left by an element belonging to $\Lambda^2 \mathcal{G} \subset Cl(\mathcal{G})$, still denoted by $\rho_*(\tilde{s}^* \alpha(X))$. Such an element satisfies

$$\tau(\rho_*(\tilde{s}^* \alpha(X))) = -\rho_*(\tilde{s}^* \alpha(X)),$$

and we have

$$\begin{aligned} \langle\langle \nabla_X \varphi, \varphi' \rangle\rangle + \langle\langle \varphi, \nabla_X \varphi' \rangle\rangle &= \tau\{[\varphi']\} (\partial_X [\varphi] + \rho_*(\tilde{s}^* \alpha(X))[\varphi]) \\ &\quad + \tau\{\partial_X [\varphi'] + \rho_*(\tilde{s}^* \alpha(X))[\varphi']\} [\varphi] \\ &= \tau\{[\varphi']\} \partial_X [\varphi] + \tau\{\partial_X [\varphi']\} [\varphi] \\ &= \partial_X \langle\langle \varphi, \varphi' \rangle\rangle. \end{aligned}$$

\square

We finally note that there is a natural action of $Spin(\mathcal{G})$ on $U\Sigma$, by right multiplication: for $\varphi = [\tilde{s}, [\varphi]] \in U\Sigma = \tilde{Q} \times_\rho Spin(\mathcal{G})$ and $a \in Spin(\mathcal{G})$ we set

$$(14) \quad \varphi \cdot a := [\tilde{s}, [\varphi] \cdot a] \in U\Sigma.$$

2.5. The spin geometry of a submanifold of G . We keep the notation of the previous section, assuming moreover here that M is a submanifold of a Lie group G and that $E \rightarrow M$ is its normal bundle. If we consider spin structures on TM and on E whose sum is the trivial spin structure of $TM \oplus E$ [15], we have

$$\Sigma = \tilde{Q} \times_{\rho} Cl(\mathcal{G}) \simeq M \times Cl(\mathcal{G}),$$

where the last bundle is the spinor bundle of G restricted to M . Two connections are thus defined on Σ , the connection ∇ and the connection ∇^G ; they satisfy the following Gauss formula:

$$(15) \quad \nabla_X^G \varphi = \nabla_X \varphi + \frac{1}{2} \sum_{j=1}^p e_j \cdot B(X, e_j) \cdot \varphi$$

for all $\varphi \in \Gamma(\Sigma)$ and all $X \in \Gamma(TM)$, where $B : TM \times TM \rightarrow E$ is the second fundamental form of M into G and e_1, \dots, e_p is an orthonormal basis of TM . We refer to [1] for the proof (in a slightly different context). Since the covariant derivative of a left invariant spinor field is given by (7), the restriction to M of such a spinor field satisfies

$$(16) \quad \nabla_X \varphi = -\frac{1}{2} \sum_{j=1}^p e_j \cdot B(X, e_j) \cdot \varphi + \frac{1}{2} \Gamma(X) \cdot \varphi$$

for all $X \in TM$.

3. MAIN RESULT

We consider a p -dimensional Riemannian manifold M and a bundle $E \rightarrow M$ of rank q , with a fibre metric and a compatible connection. We assume that E and TM are oriented and spin, with given spin structures, and consider the spinor bundles Σ and $U\Sigma$ introduced in the previous section. We suppose that a bilinear and symmetric map $B : TM \times TM \rightarrow E$ is given, and we moreover do the following two assumptions:

- (1) There exists a bundle isomorphism

$$(17) \quad f : TM \oplus E \rightarrow M \times \mathcal{G}$$

which preserves the metrics; this mapping permits to define a bundle map

$$(18) \quad \Gamma : TM \oplus E \rightarrow \Lambda^2(TM \oplus E)$$

such that, for all $X, Y \in \Gamma(TM \oplus E)$,

$$(19) \quad f(\Gamma(X)(Y)) = \Gamma(f(X))(f(Y))$$

where on the right-hand side Γ is the map defined on \mathcal{G} by (3), together with the following notion: a section $Z \in \Gamma(TM \oplus E)$ will be said to be left invariant if $f(Z) : M \rightarrow \mathcal{G}$ is a constant map.

- (2) The covariant derivative of a left invariant section $Z \in \Gamma(TM \oplus E)$ is given by

$$(20) \quad \nabla_X Z = \Gamma(X)(Z) - B(X, Z^T) + B^*(X, Z^N)$$

for all $X \in TM$, where $Z = Z^T + Z^N$ in $TM \oplus E$ and $B^* : TM \times E \rightarrow TM$ is the bilinear map such that for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(E)$

$$\langle B(X, Y), N \rangle = \langle Y, B^*(X, N) \rangle.$$

Remark 1. *These two assumptions are equivalent to the assumptions made in [11, 18]: they are necessary to write down the equations of Gauss, Codazzi and Ricci in a general metric Lie group, and to obtain a Fundamental Theorem for immersions in that context; see Section 4.*

Remark 2. *Sometimes it is convenient to write these assumptions in some local frames. For sake of simplicity, we assume that E is a trivial line bundle, oriented by a unit section ν . Let $(e_1^o, e_2^o, \dots, e_n^o)$ be an orthonormal basis of \mathcal{G} and $\Gamma_{ij}^k \in \mathbb{R}$, $1 \leq i, j, k \leq n$, be such that*

$$\Gamma(e_i^o)(e_j^o) = \sum_{k=1}^n \Gamma_{ij}^k e_k^o.$$

We set, for $i = 1, \dots, n$, $\underline{e}_i \in \Gamma(TM \oplus E)$ such that $f(\underline{e}_i) = e_i^o$, and $f_i \in C^\infty(M)$, $T_i \in \Gamma(TM)$ such that $\underline{e}_i = T_i + f_i \nu$. Since f preserves the metrics, the vectors $\underline{e}_1, \underline{e}_2, \dots, \underline{e}_n$ are orthonormal, and we have

$$(21) \quad \langle T_i, T_j \rangle + f_i f_j = \delta_{ij}$$

for all $i, j = 1, \dots, n$. The assumption (20) then reads as follows: for all $X \in TM$ and $j = 1, \dots, n$,

$$(22) \quad \nabla_X T_j = \sum_{i,k} \Gamma_{ij}^k \langle X, T_i \rangle T_k + f_j S(X),$$

$$(23) \quad df_j(X) = \sum_{i,k} \Gamma_{ij}^k f_k \langle X, T_i \rangle - h(X, T_j)$$

where $S(X) = B^(X, \nu)$ and $h(X, Y) = \langle B(X, Y), \nu \rangle$. Conversely, if vector fields $T_i \in \Gamma(TM)$ and functions $f_i \in C^\infty(M)$, $1 \leq i \leq n$, are given such that (21), (22) and (23) hold, we may define a bundle isomorphism $f : TM \oplus E \rightarrow M \times \mathcal{G}$ preserving the metrics and such that (20) holds: setting $\underline{e}_i = T_i + f_i \nu$, we define f such that $f(\underline{e}_i) = e_i^o$, $i = 1, \dots, n$.*

We state the main result of the paper:

Theorem 1. *We moreover assume that M is simply connected. The following statements are equivalent:*

- (1) *There exists a section $\varphi \in \Gamma(U\Sigma)$ such that*

$$(24) \quad \nabla_X \varphi = -\frac{1}{2} \sum_{j=1}^p e_j \cdot B(X, e_j) \cdot \varphi + \frac{1}{2} \Gamma(X) \cdot \varphi$$

for all $X \in TM$.

- (2) *There exists an isometric immersion $F : M \rightarrow G$ with normal bundle E and second fundamental form B .*

More precisely, if φ is a solution of (24), replacing φ by $\varphi \cdot a$ for some $a \in \text{Spin}(\mathcal{G})$ if necessary, and considering the \mathcal{G} -valued 1-form ξ defined by

$$(25) \quad \xi(X) := \langle X \cdot \varphi, \varphi \rangle$$

for all $X \in TM$, the formula $F = \int \xi$ defines an isometric immersion in G with normal bundle E and second fundamental form B . Here \int stands for the Darboux integral, i.e. $F = \int \xi : M \rightarrow G$ is such that $F^ \omega_G = \xi$, where $\omega_G \in \Omega^1(G, \mathcal{G})$ is the Maurer-Cartan form of G defined in (1). Reciprocally, an isometric immersion*

$M \rightarrow G$ with normal bundle E and second fundamental form B may be written in that form.

The formula $F = \int \xi$ where ξ is defined by (25) may be regarded as a generalized Weierstrass representation formula.

This theorem generalizes the main result of [4] to a Lie group equipped with a left invariant metric (see Section 5).

Remark 3. If φ is a solution of (24) and a belongs to $Spin(\mathcal{G})$, $\varphi' := \varphi \cdot a$ is also a solution of (24) (see (14) for the definition of $\varphi \cdot a$). Moreover the associated 1-forms ξ_φ and $\xi_{\varphi'}$ are linked by

$$(26) \quad \xi_{\varphi'} = \tau(a) \xi_\varphi \quad a = Ad(a^{-1}) \circ \xi_\varphi.$$

Let us recall that a 1-form $\xi \in \Omega^1(M, \mathcal{G})$ is Darboux integrable if and only if it satisfies the structure equation $d\xi + [\xi, \xi] = 0$ (M is simply connected). The theorem thus says that if φ is a solution of (24), it is possible to find an other solution φ' of this equation such that $\xi_{\varphi'}$ is Darboux integrable and $F = \int \xi_{\varphi'}$ is an immersion with normal bundle E and second fundamental form B . The proof of (1) \Rightarrow (2) in the theorem will in fact follow these lines. See also Remark 5 below.

Remark 4. Setting

$$\vec{H} = \frac{1}{2} \sum_{j=1}^p B(e_j, e_j) \in E \quad \text{and} \quad \gamma = \frac{1}{2} \sum_{j=1}^p e_j \cdot \Gamma(e_j) \in Cl(TM \oplus E)$$

where e_1, \dots, e_p is an orthonormal basis of TM , a solution φ of (24) is a solution of the Dirac equation

$$(27) \quad D\varphi := \sum_{j=1}^p e_j \cdot \nabla_{e_j} \varphi = (\vec{H} + \gamma) \cdot \varphi.$$

This equation will be especially interesting for the representation of a surface in a 3-dimensional Lie group (see Section 5).

We now prove the theorem: (1) \Rightarrow (2) will be a consequence of Propositions 3.1 and 3.2 below, and (2) \Rightarrow (1) will be proved at the end of the section.

Proposition 3.1. Assume that $\varphi \in \Gamma(U\Sigma)$ is a solution of (24) and define ξ by (25). Then

- (1) ξ takes its values in $\mathcal{G} \subset Cl(\mathcal{G})$;
- (2) there exists $T \in SO(\mathcal{G})$ such that $\xi = T \circ f$;
- (3) replacing φ by $\varphi \cdot a$ where $a \in Spin(\mathcal{G})$ is such that $Ad(a) = T$, we have $\xi = f$, and ξ satisfies the structure equation

$$(28) \quad d\xi + [\xi, \xi] = 0.$$

Proof. (1). By the very definition of ξ , we have

$$\xi(X) = \tau[\varphi][X][\varphi]$$

for all $X \in TM$, where $[X]$ and $[\varphi]$ represent X and φ in a given frame \tilde{s} of \tilde{Q} . Since $[X]$ belongs to $\mathcal{G} \subset Cl(\mathcal{G})$ and $[\varphi]$ is an element of $Spin(\mathcal{G})$, $\xi(X)$ belongs to \mathcal{G} .

(2). Let us first show that for every left invariant section $Z \in \Gamma(TM \oplus E)$, the map

$\xi(Z) : M \rightarrow \mathcal{G}$ is constant: if $Z \in \Gamma(TM \oplus E)$ is left invariant, we compute, for $X \in TM$,

$$\partial_X \xi(Z) = \langle \nabla_X Z \cdot \varphi, \varphi \rangle + \langle Z \cdot \nabla_X \varphi, \varphi \rangle + \langle Z \cdot \varphi, \nabla_X \varphi \rangle.$$

But, by (24),

$$\begin{aligned} \langle Z \cdot \nabla_X \varphi, \varphi \rangle + \langle Z \cdot \varphi, \nabla_X \varphi \rangle &= \langle [-\Gamma(X) + \sum_{j=1}^p e_j \cdot B(X, e_j), Z] \cdot \varphi, \varphi \rangle \\ (29) \qquad \qquad \qquad &= \langle \{-\Gamma(X)(Z) + B(X, Z^T) - B^*(X, Z^N)\} \cdot \varphi, \varphi \rangle \end{aligned}$$

where the brackets $[\cdot, \cdot]$ stand here for the commutator in $Cl(TM \oplus E)$ and where we use Lemmas A.1 and A.3 in the last step. Thus $\partial_X \xi(Z) = 0$ by (20), and $\xi(Z) : M \rightarrow \mathcal{G}$ is constant. Now, if (e_1^o, \dots, e_n^o) is a fixed orthonormal basis of \mathcal{G} and denoting by $\underline{e}_1, \dots, \underline{e}_n$ the left invariant sections of $TM \oplus E$ such that $f(\underline{e}_i) = e_i^o$, $i = 1, \dots, n$, we have, for all section $Z = \sum_i Z_i \underline{e}_i \in \Gamma(TM \oplus E)$,

$$\xi(Z) = \sum_{i=1}^n Z_i \xi(\underline{e}_i)$$

where $(\xi(\underline{e}_1), \dots, \xi(\underline{e}_n))$ is a constant orthonormal basis of \mathcal{G} . Considering the orthogonal transformation $T : \mathcal{G} \rightarrow \mathcal{G}$ such that $T(e_i^o) = \xi(\underline{e}_i)$, $i = 1, \dots, n$, we get

$$\xi(Z) = \sum_{i=1}^n Z_i T(e_i^o) = T \left(\sum_{i=1}^n Z_i e_i^o \right) = T(f(Z)),$$

i.e. $\xi = T \circ f$.

(3). For all $a \in Spin(\mathcal{G})$ and $X \in TM$, we have

$$\begin{aligned} \langle \langle X \cdot (\varphi \cdot a), \varphi \cdot a \rangle \rangle &= \tau([\varphi]a)[X][\varphi]a \\ &= \tau(a) \langle \langle X \cdot \varphi, \varphi \rangle \rangle a \\ &= Ad(a^{-1})(\xi(X)) \\ &= Ad(a^{-1})(T \circ f(X)); \end{aligned}$$

thus, replacing φ by $\varphi \cdot a$ where $a \in Spin(\mathcal{G})$ is such that $Ad(a) = T$ we get $\xi = f$. By the computation in (29), we have, for $X, Y \in \Gamma(TM)$ such that $\nabla X = \nabla Y = 0$ at x_0 ,

$$\begin{aligned} \partial_X \xi(Y) &= \langle \langle Y \cdot \nabla_X \varphi, \varphi \rangle \rangle + \langle \langle Y \cdot \varphi, \nabla_X \varphi \rangle \rangle \\ &= \langle \langle \{-\Gamma(X)(Y) + B(X, Y)\} \cdot \varphi, \varphi \rangle \rangle \end{aligned}$$

and thus

$$\begin{aligned} d\xi(X, Y) &= \partial_X \xi(Y) - \partial_Y \xi(X) \\ &= -\langle \langle \{\Gamma(X)(Y) - \Gamma(Y)(X)\} \cdot \varphi, \varphi \rangle \rangle \\ &= -\xi(\Gamma(X)(Y) - \Gamma(Y)(X)) \\ &= -[\xi(X), \xi(Y)], \end{aligned}$$

since $\xi = f$, Γ satisfies (19), and by (5). \square

We keep the assumption and notation of Proposition 3.1, and moreover assume that M is simply connected; we consider

$$F : M \rightarrow G$$

such that $F^*\omega_G = \xi$ (assuming that φ is chosen in such a way that ξ satisfies the structure equation (28)). The next proposition follows from the properties of the Clifford product:

Proposition 3.2. *1. The map $F : M \rightarrow G$ is an isometry.
2. The map*

$$\begin{aligned} \Phi_E : \quad E &\rightarrow F(M) \times \mathcal{G} \\ X \in E_m &\mapsto (F(m), \xi(X)) \end{aligned}$$

is an isometry between E and the normal bundle of $F(M)$ into G , preserving connections and second fundamental forms. Here, for $X \in E$, $\xi(X)$ still stands for the quantity $\langle X \cdot \varphi, \varphi \rangle$.

Proof. For $X, Y \in \Gamma(TM \oplus E)$, we have

$$\begin{aligned} \langle \xi(X), \xi(Y) \rangle &= -\frac{1}{2} (\xi(X)\xi(Y) + \xi(Y)\xi(X)) \\ &= -\frac{1}{2} (\tau[\varphi][X][\varphi]\tau[\varphi][Y][\varphi] + \tau[\varphi][Y][\varphi]\tau[\varphi][X][\varphi]) \\ &= -\frac{1}{2} \tau[\varphi]([X][Y] + [Y][X])[\varphi] \\ &= \langle X, Y \rangle, \end{aligned}$$

since $[X][Y] + [Y][X] = -2\langle [X], [Y] \rangle = -2\langle X, Y \rangle$. This implies that F is an isometry, and that Φ_E is a bundle map between E and the normal bundle of $F(M)$ into G which preserves the metrics of the fibers. Let us denote by B_F and ∇'^F the second fundamental form and the normal connection of the immersion F ; the aim is now to prove that

$$(30) \quad \xi(B(X, Y)) = B_F(\xi(X), \xi(Y)) \quad \text{and} \quad \xi(\nabla'_X N) = \nabla'_{\xi(X)} \xi(N)$$

for $X, Y \in \Gamma(TM)$ and $N \in \Gamma(E)$. First,

$$B_F(\xi(X), \xi(Y)) = (\nabla_{\xi(X)}^G \xi(Y))^N = \{\partial_X \xi(Y) + \Gamma(\xi(X))(\xi(Y))\}^N$$

where the superscript N means that we consider the component of the vector which is normal to the immersion. We fix a point $x_0 \in M$, assume that $\nabla Y = 0$ at x_0 , and compute, using (29):

$$\begin{aligned} \partial_X \xi(Y) &= \langle Y \cdot \nabla_X \varphi, \varphi \rangle + \langle Y \cdot \varphi, \nabla_X \varphi \rangle \\ &= \langle B(X, Y) \cdot \varphi, \varphi \rangle - \langle \Gamma(X)(Y) \cdot \varphi, \varphi \rangle. \end{aligned}$$

Since $\langle B(X, Y) \cdot \varphi, \varphi \rangle = \xi(B(X, Y))$ is normal to the immersion, we get

$$\{\partial_X \xi(Y)\}^N = \xi(B(X, Y)) - \langle \Gamma(X)(Y) \cdot \varphi, \varphi \rangle^N,$$

and thus

$$\begin{aligned} B_F(\xi(X), \xi(Y)) &= \xi(B(X, Y)) - \langle \Gamma(X)(Y) \cdot \varphi, \varphi \rangle^N + \Gamma(\xi(X))(\xi(Y))^N \\ &= \xi(B(X, Y)) \end{aligned}$$

since

$$\begin{aligned} \langle \Gamma(X)(Y) \cdot \varphi, \varphi \rangle &= \xi(\Gamma(X)(Y)) \\ &= f(\Gamma(X)(Y)) \\ &= \Gamma(f(X))(f(Y)) \quad (\text{by definition of } \Gamma \text{ on } TM \oplus E) \\ &= \Gamma(\xi(X))(\xi(Y)). \end{aligned}$$

We finally show the second identity in (30): we have

$$\begin{aligned}\nabla'_{\xi(X)} \xi(N) &= (\nabla_{\xi(X)}^G \xi(N))^N \\ &= (\partial_X \xi(N) + \Gamma(\xi(X))(\xi(N)))^N \\ &= \langle \nabla'_X N \cdot \varphi, \varphi \rangle^N + \langle \langle N \cdot \nabla_X \varphi, \varphi \rangle \rangle^N + \langle \langle N \cdot \varphi, \nabla_X \varphi \rangle \rangle^N \\ &\quad + \Gamma(\xi(X))(\xi(N))^N.\end{aligned}$$

The first term in the right-hand side is $\xi(\nabla'_X N)$, and we only need to show that

$$(31) \quad \langle \langle N \cdot \nabla_X \varphi, \varphi \rangle \rangle^N + \langle \langle N \cdot \varphi, \nabla_X \varphi \rangle \rangle^N + \Gamma(\xi(X))(\xi(N))^N = 0.$$

From (29), we have

$$\langle \langle N \cdot \nabla_X \varphi, \varphi \rangle \rangle + \langle \langle N \cdot \varphi, \nabla_X \varphi \rangle \rangle = -\langle \langle B^*(X, N) \cdot \varphi, \varphi \rangle \rangle - \langle \langle \Gamma(X)(N) \cdot \varphi, \varphi \rangle \rangle,$$

which gives (31) since $\langle \langle B^*(X, N) \cdot \varphi, \varphi \rangle \rangle$ is tangent to the immersion $(B^*(X, N)$ belongs to TM) and

$$\langle \langle \Gamma(X)(N) \cdot \varphi, \varphi \rangle \rangle = \Gamma(\xi(X))(\xi(N))$$

(see the first part of the proof above). \square

We finally show the converse statement (2) \Rightarrow (1): we suppose that $F : M \rightarrow G$ is an isometric immersion with normal bundle E and second fundamental form B , we consider the orthonormal frame $s_o = 1_{SO(\mathcal{G})}$ of \mathcal{G} , and the spinor frame $\tilde{s}_o = 1_{Spin(\mathcal{G})}$ (recall that $Q_G = G \times SO(\mathcal{G})$ and $\tilde{Q}_G = G \times Spin(\mathcal{G})$; see Section 2). The spinor field $\varphi = [\tilde{s}_o, 1_{Cl(\mathcal{G})}]$ satisfies (24) as a consequence of the Gauss formulas (15)-(16); moreover, its associated 1-form is, for all $X \in TM$,

$$\xi(X) = \langle \langle F_* X \cdot \varphi, \varphi \rangle \rangle = \tau[\varphi] [F_* X] [\varphi] = [F_* X],$$

where $[F_* X] \in \mathcal{G}$ represents $F_* X$ in s_o , that is $[F_* X] = \omega_G(F_* X)$ ($\omega_G \in \Omega^1(G, \mathcal{G})$ is the Maurer-Cartan form of G). Thus $\xi = F^* \omega_G$, that is $F = \int \xi$.

Remark 5. We proved in Proposition 3.2 that if $\varphi \in \Gamma(U\Sigma)$ is a solution of (24) such that ξ_φ satisfies the structure equation (28) then $F = \int \xi_\varphi$ is an immersion with normal bundle E and second fundamental form B . By (26) it is clear that if $a \in Spin(\mathcal{G})$ is such that $Ad(a^{-1}) : \mathcal{G} \rightarrow \mathcal{G} \in SO(\mathcal{G})$ is an automorphism of Lie algebra, then $\xi_{\varphi \cdot a}$ satisfies the structure equation too; in fact, the corresponding immersions $F_\varphi = \int \xi_\varphi$ and $F_{\varphi \cdot a} = \int \xi_{\varphi \cdot a}$ are linked by the following formula: if $\Phi_a : G \rightarrow G$ is the automorphism of G such that $d(\Phi_a)_e = Ad(a^{-1})$, then Φ_a is also an isometry for the left invariant metric, and

$$(32) \quad F_{\varphi \cdot a} = L_b \circ \Phi_a \circ F_\varphi$$

for some b belonging to G . This relies on the following formula: if $\Phi : G \rightarrow G$ is an automorphism, $\omega_G \in \Omega^1(G, \mathcal{G})$ is the Maurer-Cartan form of G and $F : M \rightarrow G$ is a smooth map, then

$$(\Phi \circ F)^* \omega_G = d(\Phi)_e \circ (F^* \omega_G).$$

This formula applied to $\Phi = \Phi_a$ and $F = F_\varphi$ shows that $\Phi_a \circ F_\varphi$ is a solution of the Darboux equation associated to the form $\xi_{\varphi \cdot a}$; thus, by uniqueness of a solution of the Darboux equation, (32) holds for some b belonging to G .

4. AN APPLICATION: THE FUNDAMENTAL THEOREM FOR IMMERSIONS IN A METRIC LIE GROUP

We now show that the equations of Gauss, Ricci and Codazzi on B are exactly the integrability conditions of (24). We recall these equations for immersions in the metric Lie group G : if R^G denotes the curvature tensor of $(G, \langle \cdot, \cdot \rangle)$, and if R^T and R^N stand for the curvature tensors of the connections on TM and on E (M is a submanifold of G and E is its normal bundle), then we have, for all $X, Y, Z \in \Gamma(TM)$ and $N \in \Gamma(E)$,

(1) the Gauss equation

$$(33) \quad (R^G(X, Y)Z)^T = R^T(X, Y)Z - B^*(X, B(Y, Z)) + B^*(Y, B(X, Z)),$$

(2) the Ricci equation

$$(34) \quad (R^G(X, Y)N)^N = R^N(X, Y)N - B(X, B^*(Y, N)) + B(Y, B^*(X, N)),$$

(3) the Codazzi equation

$$(35) \quad (R^G(X, Y)Z)^N = \tilde{\nabla}_X B(Y, Z) - \tilde{\nabla}_Y B(X, Z);$$

in the last equation, $\tilde{\nabla}$ denotes the natural connection on $T^*M \otimes T^*M \otimes E$.

These equations make sense if M is an abstract manifold and $E \rightarrow M$ is an abstract bundle as in Section 3, if we assume the existence of the bundle map f in (17), since f permits to define Γ on $TM \oplus E$ by (19), and R^G may be written in terms of Γ only (see (5)-(6)). We prove the following:

Proposition 4.1. *We assume that M is simply connected. There exists $\varphi \in \Gamma(U\Sigma)$ solution of (24) if and only if $B : TM \times TM \rightarrow E$ satisfies the Gauss, Ricci and Codazzi equations.*

Proof. We first prove that the Gauss, Ricci and Codazzi equations are necessary if we have a non-trivial solution of (24). We assume that $\varphi \in \Gamma(U\Sigma)$ is a solution of (24) and compute the curvature

$$R(X, Y)\varphi = \nabla_X \nabla_Y \varphi - \nabla_Y \nabla_X \varphi - \nabla_{[X, Y]}\varphi.$$

We fix a point $x_0 \in M$, and assume that $\nabla X = \nabla Y = 0$ at x_0 . We have

$$\begin{aligned} \nabla_X \nabla_Y \varphi &= -\frac{1}{2} \sum_{j=1}^p e_j \cdot \left(\tilde{\nabla}_X B(Y, e_j) \cdot \varphi + B(Y, e_j) \cdot \nabla_X \varphi \right) \\ &\quad + \frac{1}{2} (\nabla_X \Gamma(Y) \cdot \varphi + \Gamma(Y) \cdot \nabla_X \varphi) \\ &= -\frac{1}{2} \sum_{j=1}^p e_j \cdot \tilde{\nabla}_X B(Y, e_j) \cdot \varphi - \frac{1}{4} \sum_{j,k=1}^p e_j \cdot e_k \cdot B(Y, e_j) \cdot B(X, e_k) \\ &\quad - \frac{1}{4} \sum_{j=1}^p e_j \cdot B(Y, e_j) \cdot \Gamma(X) \cdot \varphi + \frac{1}{2} \nabla_X \Gamma(Y) \cdot \varphi - \frac{1}{4} \Gamma(Y) \cdot \sum_{j=1}^p e_j \cdot B(X, e_j) \cdot \varphi \\ &\quad + \frac{1}{4} \Gamma(Y) \cdot \Gamma(X) \cdot \varphi. \end{aligned}$$

Thus

$$\begin{aligned}
 R(X, Y)\varphi &= -\frac{1}{2} \sum_{j=1}^p e_j \cdot \left(\tilde{\nabla}_X B(Y, e_j) - \tilde{\nabla}_Y B(X, e_j) \right) \cdot \varphi \\
 (36) \quad &+ \underbrace{\frac{1}{4} \sum_{j \neq k} e_j \cdot e_k \cdot (B(X, e_j) \cdot B(Y, e_k) - B(Y, e_j) \cdot B(X, e_k))}_{\mathcal{A}} \cdot \varphi \\
 &- \underbrace{\frac{1}{4} \sum_{j=1}^p (B(X, e_j) \cdot B(Y, e_j) - B(Y, e_j) \cdot B(X, e_j))}_{\mathcal{B}} \cdot \varphi \\
 &+ \underbrace{\frac{1}{2} \left[\sum_{j=1}^p e_j \cdot B(X, e_j), \Gamma(Y) \right]}_{\mathcal{C}_1} \cdot \varphi + \underbrace{-\frac{1}{2} \left[\sum_{j=1}^p e_j \cdot B(Y, e_j), \Gamma(X) \right]}_{\mathcal{C}_2} \cdot \varphi \\
 &+ \underbrace{\frac{1}{2} (\nabla_X \Gamma(Y) - \nabla_Y \Gamma(X))}_{\mathcal{C}_3} \cdot \varphi + \underbrace{-\frac{1}{2} [\Gamma(X), \Gamma(Y)]}_{\mathcal{C}_4} \cdot \varphi
 \end{aligned}$$

where the brackets stand for the commutator in the Clifford bundle $Cl(TM \oplus E)$:
 $\forall \eta, \xi \in Cl(TM \oplus E)$,

$$[\eta, \xi] = \frac{1}{2} (\eta \cdot \xi - \xi \cdot \eta).$$

We computed the second and the third terms in [4]; we only recall the result here:

Lemma 4.2. [4] *We have*

$$\mathcal{A} = \frac{1}{2} \sum_{j < k} \{ \langle B^*(X, B(Y, e_j)), e_k \rangle - \langle B^*(Y, B(X, e_j)), e_k \rangle \} e_j \cdot e_k$$

and

$$\mathcal{B} = \frac{1}{2} \sum_{k < l} \langle B(X, B^*(Y, n_k)) - B(Y, B^*(X, n_k)), n_l \rangle n_k \cdot n_l,$$

where e_1, \dots, e_p and n_1, \dots, n_q are orthonormal bases of TM and E .

We now compute the other terms in (36). We first compute the covariant derivative of Γ , considering Γ as a map

$$\Gamma : TM \oplus E \rightarrow End(TM \oplus E).$$

Lemma 4.3. *If $X, Y \in TM$ and $Z \in TM \oplus E$,*

$$\begin{aligned}
 (\nabla_X \Gamma)(Y)Z &= \{ \Gamma(X) \circ \Gamma(Y) - \Gamma(Y) \circ \Gamma(X) \} (Z) - \Gamma(\Gamma(X)Y)(Z) + \Gamma(B(X, Y))(Z) \\
 &\quad - B(X, (\Gamma(Y)Z)^T) + B^*(X, (\Gamma(Y)Z)^N) + \Gamma(Y)(B(X, Z^T) - B^*(X, Z^N)).
 \end{aligned}$$

Proof. Since the expression is tensorial, we may assume that $X, Y, Z \in \Gamma(TM \oplus E)$ are left invariant vector fields. By definition,

$$(37) \quad (\nabla_X \Gamma)(Y)Z = \nabla_X(\Gamma(Y)Z) - \Gamma(\nabla_X Y)Z - \Gamma(Y)(\nabla_X Z).$$

Since X, Y and Z are left invariant vector fields, so are $\Gamma(Y)Z$, $\nabla_X Y$ and $\nabla_X Z$, and, by (20),

$$\nabla_X(\Gamma(Y)Z) = \Gamma(X)(\Gamma(Y)Z) - B(X, (\Gamma(Y)Z)^T) + B^*(X, (\Gamma(Y)Z)^N),$$

$$\Gamma(Y)(\nabla_X Z) = \Gamma(Y)(\Gamma(X)Z) - \Gamma(Y)B(X, Z^T) + \Gamma(Y)B^*(X, Z^N)$$

and

$$\Gamma(\nabla_X Y)(Z) = \Gamma(\Gamma(X)Y)Z - \Gamma(B(X, Y^T))Z + \Gamma(B^*(X, Y^N))Z.$$

Plugging these formulas in (37) and using finally that Y belongs to TM (ie $Y^T = Y$ and $Y^N = 0$), we get the result. \square

We now regard Γ as a map

$$\Gamma : TM \oplus E \rightarrow \Lambda^2(TM \oplus E) \subset Cl(TM \oplus E),$$

and compute the term \mathcal{C}_3 in (36). According to Lemma A.1, for all $X, Y \in TM \oplus E$,

$$\Gamma(X)(Y) = [\Gamma(X), Y].$$

Lemma 4.4. *If $X, Y \in TM$,*

$$\begin{aligned} \frac{1}{2}((\nabla_X \Gamma)(Y) - (\nabla_Y \Gamma)(X)) &= [\Gamma(X), \Gamma(Y)] - \frac{1}{2}\Gamma([\Gamma(X), Y] - [\Gamma(Y), X]) \\ &\quad - \frac{1}{2} \left[\sum_{j=1}^p e_j \cdot B(X, e_j), \Gamma(Y) \right] + \frac{1}{2} \left[\sum_{j=1}^p e_j \cdot B(Y, e_j), \Gamma(X) \right]. \end{aligned}$$

Here the brackets stand for the commutator in $Cl(TM \oplus E)$.

Proof. By Lemmas A.1 and A.2 in the appendix, the linear maps $\Gamma(X) \circ \Gamma(Y) - \Gamma(Y) \circ \Gamma(X)$, $Z \mapsto \Gamma(\Gamma(X)Y)Z$ and $Z \mapsto \Gamma(B(X, Y))Z$ appearing in Lemma 4.3 are respectively represented by the bivectors $[\Gamma(X), \Gamma(Y)]$, $\Gamma([\Gamma(X), Y])$ and $\Gamma(B(X, Y))$. Moreover, by Lemma A.4 applied to the linear maps $B(X, \cdot) : TM \rightarrow E$ and $\Gamma(Y) : TM \oplus E \rightarrow TM \oplus E$, the map

$$Z \mapsto -B^*(X, (\Gamma(Y)Z)^N) + \Gamma(Y)(B^*(X, Z^N)) + B(X, (\Gamma(Y)Z)^T) - \Gamma(Y)(B(X, Z^T))$$

is represented by the bivector

$$\left[\sum_{j=1}^p e_j \cdot B(X, e_j), \Gamma(Y) \right] \in Cl(TM \oplus E).$$

The result follows. \square

We readily deduce the sum of the last four terms in (36):

Lemma 4.5. *Let us set, for $X, Y \in TM$,*

$$R^G(X, Y) = [\Gamma(X), \Gamma(Y)] - \Gamma\{[\Gamma(X), Y] - [\Gamma(Y), X]\} \in \Lambda^2(TM \oplus E),$$

the curvature tensor of G , pulled-back to $TM \oplus E$ by the bundle isomorphism f introduced in (17). Then

$$\mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4 = \frac{1}{2}R^G(X, Y).$$

We thus get from (36) the formula

$$\begin{aligned} (38) \quad R(X, Y)\varphi &= -\frac{1}{2} \sum_{j=1}^p e_j \cdot \left(\tilde{\nabla}_X B(Y, e_j) - \tilde{\nabla}_Y B(X, e_j) \right) \cdot \varphi \\ &\quad + \mathcal{A} \cdot \varphi + \mathcal{B} \cdot \varphi + \frac{1}{2}R^G(X, Y) \cdot \varphi \end{aligned}$$

where \mathcal{A} and \mathcal{B} are computed in Lemma 4.2 and R^G may be conveniently written in the form

$$\begin{aligned} R^G(X, Y) &= \sum_{1 \leq j < k \leq p} \langle R^G(X, Y)(e_j), e_k \rangle e_j \cdot e_k \\ &\quad + \sum_{j=1}^p \sum_{r=1}^q \langle R^G(X, Y)(e_j), n_r \rangle e_j \cdot n_r \\ &\quad + \sum_{1 \leq r < s \leq q} \langle R^G(X, Y)(n_r), n_s \rangle n_r \cdot n_s. \end{aligned}$$

On the other hand, the curvature of the spinorial connection is given by

$$(39) \quad \begin{aligned} R(X, Y)\varphi &= \frac{1}{2} \left(\sum_{1 \leq j < k \leq p} \langle R^T(X, Y)(e_j), e_k \rangle e_j \cdot e_k \right. \\ &\quad \left. + \sum_{1 \leq r < s \leq q} \langle R^N(X, Y)(n_r), n_s \rangle n_r \cdot n_s \right) \cdot \varphi. \end{aligned}$$

We now compare the expressions (38) and (39): since in a given frame \tilde{s} belonging to \tilde{Q} , φ is represented by an element which is invertible in $Cl(\mathcal{G})$ (it is in fact represented by an element belonging to $Spin(\mathcal{G})$), we may identify the coefficients and get

$$\begin{aligned} \langle R^T(X, Y)(e_j), e_k \rangle &= \langle B^*(X, B(Y, e_j)), e_k \rangle - \langle B^*(Y, B(X, e_j)), e_k \rangle + \langle R^G(X, Y)(e_j), e_k \rangle, \\ \langle R^N(X, Y)(n_r), n_s \rangle &= \langle B(X, B^*(Y, n_r)), n_s \rangle - \langle B(Y, B^*(X, n_r)), n_s \rangle + \langle R^G(X, Y)(n_r), n_s \rangle \end{aligned}$$

and

$$\langle \tilde{\nabla}_X B(Y, e_j) - \tilde{\nabla}_Y B(X, e_j), n_r \rangle = \langle R^G(X, Y)(e_j), n_r \rangle$$

for all the indices. These equations are the equations of Gauss, Ricci and Codazzi.

We now prove that the equations of Gauss, Ricci and Codazzi are also sufficient to get a solution of (24). The calculations above in fact show that the connection on Σ defined by

$$(40) \quad \nabla'_X \varphi := \nabla_X \varphi + \frac{1}{2} \sum_{j=1}^p e_j \cdot B(X, e_j) \cdot \varphi - \frac{1}{2} \Gamma(X) \cdot \varphi$$

for all $\varphi \in \Gamma(\Sigma)$ and $X \in \Gamma(TM)$ is flat if and only if the equations of Gauss, Ricci and Codazzi hold. But if this connection is flat there exists a solution $\varphi \in \Gamma(U\Sigma)$ of (24); this is because ∇' may be also interpreted as a connection on $U\Sigma$ regarded as a principal bundle (of group $Spin(\mathcal{G})$, acting on the right): indeed, ∇ defines such a connection (since it comes from a connection on \tilde{Q}), and the right hand side term in (40) defines a linear map

$$\begin{aligned} TM &\rightarrow \chi_V^{inv}(U\Sigma) \\ X &\mapsto \varphi \mapsto \frac{1}{2} \sum_{j=1}^p e_j \cdot B(X, e_j) \cdot \varphi - \frac{1}{2} \Gamma(X) \cdot \varphi \end{aligned}$$

from TM to the vector fields on $U\Sigma$ which are vertical and invariant under the action of the group (these vector fields are of the form $\varphi \mapsto \eta \cdot \varphi$, $\eta \in \Lambda^2(TM \oplus E) \subset$

$Cl(TM \oplus E)$). Assuming that the equations of Gauss, Codazzi and Ricci hold, we thus get a solution $\varphi \in \Gamma(U\Sigma)$ of (24). \square

The considerations above give a spinorial proof of the Fundamental Theorem of submanifold theory in the metric Lie group G (see [18] for another proof). We keep the hypotheses and notation of the beginning of Section 3.

Corollary 1. *We moreover assume that M is simply connected and that $B : TM \times TM \rightarrow E$ satisfies the equations of Gauss, Codazzi and Ricci (33)-(35). Then there is an isometric immersion of M into G with normal bundle E and second fundamental form B . The immersion is unique up to a rigid motion in G , that is up to a transformation of the form*

$$(41) \quad \begin{aligned} L_b \circ \Phi_a : \quad G &\rightarrow G \\ g &\mapsto b\Phi_a(g) \end{aligned}$$

where $a \in Spin(\mathcal{G})$ is such that $Ad(a) : \mathcal{G} \rightarrow \mathcal{G}$ is an automorphism of Lie algebra, $\Phi_a : G \rightarrow G$ is the group automorphism such that $d(\Phi_a)_e = Ad(a)$, and b belongs to G .

Proof. The equations of Gauss, Codazzi and Ricci are the integrability conditions of (24). We thus get a solution $\varphi \in \Gamma(U\Sigma)$ of (24); with such a spinor field at hand, $F = \int \xi$ where ξ is defined in (25) is the immersion. Finally, a solution of (24) is unique up to the right action of an element of $Spin(\mathcal{G})$; the right multiplication of φ by $a \in Spin(\mathcal{G})$ and the left multiplication by $b \in G$ in the last integration give also an immersion, if $Ad(a) : \mathcal{G} \rightarrow \mathcal{G}$ is moreover an automorphism of Lie algebra. This immersion is obtained from the immersion defined by φ by a rigid motion, as described in (41). \square

Remark 6. *In \mathbb{R}^n , a rigid motion as in (41) is a transformation of the form*

$$\begin{aligned} \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ x &\mapsto ax + b, \end{aligned}$$

with $a \in SO(n)$ and $b \in \mathbb{R}^n$.

5. SPECIAL CASES

5.1. Submanifolds in \mathbb{R}^n . If the metric Lie group is \mathbb{R}^n with its natural metric, we recover the main result of [4]. We suppose that M is a p -dimensional Riemannian manifold, $E \rightarrow M$ a bundle of rank q , with a fibre metric and a compatible connection. We assume that TM and E are oriented and spin with given spin structures, and that $B : TM \times TM \rightarrow E$ is bilinear and symmetric.

Theorem 2. [4] *We moreover assume that M is simply connected. The following statements are equivalent:*

(1) *There exists a section $\varphi \in \Gamma(U\Sigma)$ such that*

$$(42) \quad \nabla_X \varphi = -\frac{1}{2} \sum_{j=1}^p e_j \cdot B(X, e_j) \cdot \varphi$$

for all $X \in TM$.

- (2) *There exists an isometric immersion $F : M \rightarrow \mathbb{R}^n$ with normal bundle E and second fundamental form B .*

Moreover, $F = \int \xi$ where ξ is the \mathbb{R}^n -valued 1-form defined by

$$(43) \quad \xi(X) := \langle X \cdot \varphi, \varphi \rangle$$

for all $X \in TM$.

Proof. We only prove (1) \Rightarrow (2). This will be a consequence of Theorem 1 if we may define a bundle map f as in (17) such that (20) holds. We assume that φ is a solution of (42), and set

$$\begin{aligned} f : TM \oplus E &\rightarrow M \times \mathbb{R}^n \\ Z &\mapsto \langle Z \cdot \varphi, \varphi \rangle. \end{aligned}$$

The map Γ defined by (19) is $\Gamma = 0$. We now show that (20) is satisfied for every $Z \in \Gamma(TM \oplus E)$ such that $f(Z) : M \rightarrow \mathbb{R}^n$ is a constant map: for all $X \in TM$, we have $\partial_X \{f(Z)\} = 0$, which reads

$$\langle \nabla_X Z \cdot \varphi, \varphi \rangle + \langle Z \cdot \nabla_X \varphi, \varphi \rangle + \langle Z \cdot \varphi, \nabla_X \varphi \rangle = 0.$$

But (42) gives

$$\langle Z \cdot \nabla_X \varphi, \varphi \rangle + \langle Z \cdot \varphi, \nabla_X \varphi \rangle = \langle \{B(X, Z^T) - B^*(X, Z^N)\} \cdot \varphi, \varphi \rangle$$

(see the computations in (29) with $\Gamma = 0$). Thus

$$\langle \nabla_X Z \cdot \varphi, \varphi \rangle = \langle \{-B(X, Z^T) + B^*(X, Z^N)\} \cdot \varphi, \varphi \rangle$$

and

$$\nabla_X Z = -B(X, Z^T) + B^*(X, Z^N),$$

which is (20) with $\Gamma = 0$. \square

5.2. Submanifolds in \mathbb{H}^n . Spinor representations of submanifolds in \mathbb{H}^n with its natural metric were already given in [16, 3, 4]. We give here another representation using the group structure of \mathbb{H}^n , with an arbitrary left invariant metric. Let us set

$$\mathbb{H}^n = \{a = (a', a_n) \in \mathbb{R}^n : a_n > 0\},$$

and, for $a \in \mathbb{H}^n$, the transformation

$$\begin{aligned} \varphi_a : \mathbb{R}^{n-1} &\rightarrow \mathbb{R}^{n-1} \\ x &\mapsto a_n x + a'; \end{aligned}$$

φ_a is an homothety composed by a translation. The homotheties composed by translations naturally form a group under composition, and the bijection

$$\begin{aligned} \varphi : \mathbb{H}^n &\rightarrow \{\text{homotheties-translations } \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}\} \\ a &\mapsto \varphi_a \end{aligned}$$

induces a group structure on \mathbb{H}^n : it is such that

$$(44) \quad ab = (a_n b' + a', a_n b_n)$$

for all $a, b \in \mathbb{H}^n$; the identity element is $e = (0, 1) \in \mathbb{H}^n$. Let us denote by $(e_1^o, e_2^o, \dots, e_n^o)$ the canonical basis of $T_e \mathbb{H}^n = \mathbb{R}^n$ and keep the same letters to denote the corresponding left invariant vector fields on \mathbb{H}^n . The Lie bracket may be easily seen to be given by

$$[e_i^o, e_j^o] = 0 \quad \text{and} \quad [e_n^o, e_i^o] = e_i^o$$

for $i, j = 1, \dots, n-1$. This may also be written in the form

$$(45) \quad [X, Y] = l(X)Y - l(Y)X$$

for all $X, Y \in \mathbb{R}^n$, where $l : \mathbb{R}^n \rightarrow \mathbb{R}$ is the linear form such that $l(e_i^o) = 0$ if $i \leq n-1$ and $l(e_n^o) = 1$. This property implies that every left invariant metric on \mathbb{H}^n has constant negative curvature $-|l|^2$ [14, 13].

We suppose that a left invariant metric $\langle \cdot, \cdot \rangle$ is given on \mathbb{H}^n , and consider the vector $U_o \in T_e \mathbb{H}^n$ such that $l(X) = \langle U_o, X \rangle$ for all $X \in T_e \mathbb{H}^n$. We have $|U_o| = |l|$, and, by the Koszul formula (4),

$$(46) \quad \Gamma(X)(Y) = -\langle Y, U_o \rangle X + \langle X, Y \rangle U_o$$

for all $X, Y \in T_e \mathbb{H}^n$.

We keep the hypotheses made at the beginning of Section 5.1. We suppose moreover that $U \in \Gamma(TM \oplus E)$ is given such that $|U| = |l|$ and, for all $X \in TM$,

$$(47) \quad \nabla_X U = -|U|^2 X + \langle X, U \rangle U - B(X, U^T) + B^*(X, U^N).$$

We set, for $X \in TM$ and $Y \in TM \oplus E$,

$$(48) \quad \Gamma(X)(Y) = -\langle Y, U \rangle X + \langle X, Y \rangle U.$$

Remark 7. Equation (47) implies the following:

- (1) U is a solution of (20), with the definition (48) of Γ .
- (2) The norm of U is constant, since, by a straightforward computation,

$$d|U|^2(X) = 2\langle \nabla_X U, U \rangle = 0$$

for all $X \in TM$. The additional hypothesis $|U| = |l|$ is thus not very restrictive.

We note that it is not necessary to assume the existence of U solution of (47) to get a spinor representation of a submanifold in \mathbb{H}^n if \mathbb{H}^n is regarded as the set of unit vectors in Minkowski space $\mathbb{R}^{n,1}$ [16, 3, 4]. Nevertheless, this hypothesis seems necessary if we consider \mathbb{H}^n as a group, since the group structure introduces an anisotropy: the vector $e_n \in T_e \mathbb{H}^n$ is indeed a special direction for the group structure.

Let us construct the spinor bundles Σ and $U\Sigma$ on M as in Section 2.4 with here $\mathcal{G} = T_e \mathbb{H}^n$.

Theorem 3. We assume that M is simply connected. The following statements are equivalent:

- (1) There exists a spinor field $\varphi \in \Gamma(U\Sigma)$ solution of (24) where Γ is defined by (48).
- (2) There exists an isometric immersion $M \rightarrow \mathbb{H}^n$ with normal bundle E and second fundamental form B .

Proof. We assume that $\varphi \in \Gamma(U\Sigma)$ is a solution of (24) where Γ is defined by (48), and define $f : TM \oplus E \rightarrow M \times T_e \mathbb{H}^n$ by

$$f(Z) = \langle \langle Z \cdot \varphi, \varphi \rangle \rangle$$

for all $Z \in TM \oplus E$. Let us first observe that if Z is a vector field solution of (20), then $f(Z)$ is constant: we have, for all $X \in TM$,

$$\partial_X f(Z) = \langle \langle \nabla_X Z \cdot \varphi, \varphi \rangle \rangle + \langle \langle Z \cdot \nabla_X \varphi, \varphi \rangle \rangle + \langle \langle Z \cdot \varphi, \nabla_X \varphi \rangle \rangle;$$

this is 0, by (20), (24) and the computation (29). Since U is a solution of (20) (see Remark 7), we deduce that $f(U) \in T_e \mathbb{H}^n$ is a constant, and, since $|f(U)| = |U| = |U_o|$, replacing φ by $\varphi \cdot a$ for some $a \in Spin(T_e \mathbb{H}^n)$ if necessary, we may suppose that $f(U) = U_o$. Since Γ is defined on $T_e \mathbb{H}^n$ by (46) and on $TM \oplus E$ by (48), and since f preserves the metrics, it is straightforward to see that $f(\Gamma(X)(Y)) = \Gamma(f(X))(f(Y))$ for all $X, Y \in TM \oplus E$. Finally, (20) holds for all $Z \in \Gamma(TM \oplus E)$ such that $f(Z)$ is constant: this is the same argument as in the proof of Theorem 2 in Section 5.1, just adding the term Γ . The result then follows from Theorem 1. \square

5.3. Hypersurfaces in a metric Lie group. We assume that G is a simply connected n -dimensional metric Lie group, M is a p -dimensional Riemannian manifold, $n = p+1$, and E is the trivial line bundle on M , oriented by a unit section $\nu \in \Gamma(E)$. We moreover suppose that M is simply connected and that $h : TM \times TM \rightarrow \mathbb{R}$ is a given symmetric bilinear form, and that the hypotheses (1) and (2) of Section 3 with $B = h\nu$ hold. According to Theorem 1, an isometric immersion of M into G with second fundamental form h is equivalent to a section φ of $\Gamma(U\Sigma)$ solution of the Killing equation (24). Note that $Q_E \simeq M$ and the double covering $\tilde{Q}_E \rightarrow Q_E$ is trivial, since M is assumed to be simply connected. Fixing a section \tilde{s}_E of \tilde{Q}_E we get an injective map

$$\begin{aligned} \tilde{Q}_M &\rightarrow \tilde{Q}_M \times_M \tilde{Q}_E =: \tilde{Q} \\ \tilde{s}_M &\mapsto (\tilde{s}_M, \tilde{s}_E). \end{aligned}$$

Using

$$Cl_p \simeq Cl_{p+1}^0 \subset Cl_{p+1}$$

(induced by the Clifford map $\mathbb{R}^p \rightarrow Cl_{p+1}$, $X \mapsto X \cdot e_{p+1}$), we deduce a bundle isomorphism

$$(49) \quad \begin{aligned} \tilde{Q}_M \times_\rho Cl_p &\rightarrow \tilde{Q} \times_\rho Cl_{p+1}^0 \subset \Sigma \\ \psi &\mapsto \psi^*. \end{aligned}$$

It satisfies the following properties: for all $X \in TM$ and $\psi \in \tilde{Q}_M \times_\rho Cl_p$,

$$(50) \quad (X \cdot \psi)^* = X \cdot \nu \cdot \psi^* \quad \text{and} \quad \nabla_X(\psi^*) = (\nabla_X \psi)^*.$$

To write down the Killing equation (24) in the bundle $\tilde{Q}_M \times_\rho Cl_p$, we need to decompose the Clifford action of $\Gamma(X)$ into its tangent and its normal parts:

Lemma 5.1. *Recall the notation introduced in Remark 2. Then, for all $X \in TM$,*

$$(51) \quad \Gamma(X) = \sum_{i=1}^n \langle X, T_i \rangle \sum_{1 \leq j < k \leq n} \Gamma_{ij}^k \left(\frac{1}{2} (T_j \cdot T_k - T_k \cdot T_j) + (f_k T_j - f_j T_k) \cdot \nu \right).$$

Proof. We have

$$X = \sum_{i=1}^n \langle X, \underline{e}_i \rangle \underline{e}_i = \sum_{i=1}^n \langle X, T_i \rangle \underline{e}_i,$$

$$\begin{aligned}
\Gamma(X)(\underline{e}_j) &= \sum_{i=1}^n \langle X, T_i \rangle \Gamma(\underline{e}_i)(\underline{e}_j) \\
&= \sum_{i=1}^n \langle X, T_i \rangle \sum_{k=1}^n \Gamma_{ij}^k \underline{e}_k \\
&= \sum_{1 \leq i, k \leq n} \Gamma_{ij}^k \langle X, T_i \rangle (T_k + f_k \nu),
\end{aligned}$$

and thus

$$\begin{aligned}
\Gamma(X) &= \frac{1}{2} \sum_{j=1}^n \underline{e}_j \cdot \Gamma(X)(\underline{e}_j) \\
&= \frac{1}{2} \sum_{j=1}^n (T_j + f_j \nu) \cdot \sum_{1 \leq i, k \leq n} \Gamma_{ij}^k \langle X, T_i \rangle (T_k + f_k \nu) \\
&= \frac{1}{2} \sum_{1 \leq i, j, k \leq n} \Gamma_{ij}^k \langle X, T_i \rangle (T_j + f_j \nu) \cdot (T_k + f_k \nu).
\end{aligned}$$

Now

$$(T_j + f_j \nu) \cdot (T_k + f_k \nu) = T_j \cdot T_k + f_k T_j \cdot \nu - f_j T_k \cdot \nu - f_j f_k,$$

and the result follows since $\Gamma_{ij}^k = -\Gamma_{ik}^j$. \square

The section $\varphi \in \Gamma(U\Sigma)$ solution of (24) thus identifies to a section ψ of $\tilde{Q}_M \times_\rho Cl_p$ solution of

$$\begin{aligned}
\nabla_X \psi &= -\frac{1}{2} \sum_{j=1}^p h(X, e_j) e_j \cdot \psi + \frac{1}{2} \tilde{\Gamma}(X) \cdot \psi \\
&= -\frac{1}{2} S(X) \cdot \psi + \frac{1}{2} \tilde{\Gamma}(X) \cdot \psi
\end{aligned}$$

for all $X \in TM$, where

$$(52) \quad \tilde{\Gamma}(X) = \sum_{i=1}^n \langle X, T_i \rangle \sum_{1 \leq j < k \leq n} \Gamma_{ij}^k \left(\frac{1}{2} (T_j \cdot T_k - T_k \cdot T_j) + (f_k T_j - f_j T_k) \right)$$

and $S : TM \rightarrow TM$ is the symmetric operator associated to h . We deduce the following result:

Theorem 4. *Let $S : TM \rightarrow TM$ be a symmetric operator. The following two statements are equivalent:*

- (1) *there exists an isometric immersion of M into G with shape operator S ;*
- (2) *there exists a normalized spinor field $\psi \in \Gamma(\tilde{Q}_M \times_\rho Cl_p)$ solution of*

$$(53) \quad \nabla_X \psi = -\frac{1}{2} S(X) \cdot \psi + \frac{1}{2} \tilde{\Gamma}(X) \cdot \psi$$

for all $X \in TM$, where $\tilde{\Gamma}$ is defined in (52).

Here, a spinor field $\psi \in \Gamma(\tilde{Q}_M \times_\rho Cl_p)$ is said to be normalized if it is represented in some frame $\tilde{s} \in \tilde{Q}_M$ by an element $[\psi] \in Cl_p \simeq Cl_{p+1}^0$ belonging to $Spin(p+1)$.

We will see below explicit representation formulas in the cases of the dimensions 3 and 4.

5.4. Surfaces in a 3-dimensional metric Lie group. Since $Cl_2 \simeq \Sigma_2$ we have

$$\tilde{Q}_M \times_\rho Cl_2 \simeq \Sigma M,$$

and φ is equivalent to a spinor field $\psi \in \Gamma(\Sigma M)$ solution of (53) and such that $|\psi| = 1$. Moreover, the explicit representation formula $F = \int \xi$ may be written in terms of ψ : it may be proved by a computation that

$$(54) \quad \langle \langle X \cdot \varphi, \varphi \rangle \rangle = i2\mathcal{R}e \langle X \cdot \psi^+, \psi^- \rangle + j (\langle X \cdot \psi^+, \alpha(\psi^+) \rangle - \langle X \cdot \psi^-, \alpha(\psi^-) \rangle)$$

where the brackets $\langle \cdot, \cdot \rangle$ stand here for the natural hermitian product on Σ_2 and $\alpha : \Sigma_2 \rightarrow \Sigma_2$ is the natural quaternionic structure. If $G = \mathbb{R}^3$, this is the explicit representation formula given in [8] (see also [3]).

We also note that the expression (52) of $\tilde{\Gamma}$ simplifies if the Lie group is 3-dimensional:

Lemma 5.2. *If $j, k, j \neq k$, belong to $\{1, 2, 3\}$, let us denote by $l_{jk} \in \{1, 2, 3\}$ the number such that (j, k, l_{jk}) is a permutation of $\{1, 2, 3\}$ and by $\epsilon_{jk} = \pm 1$ the sign of this permutation. Then, for all $X \in TM$,*

$$\tilde{\Gamma}(X) = \sum_{i=1}^3 \langle X, T_i \rangle \sum_{1 \leq j < k \leq 3} \Gamma_{ij}^k \epsilon_{jk} (f_{l_{jk}} - T_{l_{jk}}) \cdot \omega$$

where $\omega \in Cl(TM)$ is the area element of M .

Proof. Keeping the notation introduced above, we note that

$$\underline{e}_j \cdot \underline{e}_k \cdot \underline{e}_{l_{jk}} = \epsilon_{jk} \omega \cdot \nu,$$

which yields

$$\underline{e}_j \cdot \underline{e}_k = -\epsilon_{jk} \omega \cdot \nu \cdot \underline{e}_{l_{jk}}.$$

Thus

$$\begin{aligned} T_j \cdot T_k + (f_k T_j - f_j T_k) \cdot \nu - f_j f_k &= -\epsilon_{jk} \omega \cdot \nu \cdot (T_{l_{jk}} + f_{l_{jk}} \nu) \\ &= \epsilon_{jk} (f_{l_{jk}} - T_{l_{jk}} \cdot \nu) \cdot \omega \end{aligned}$$

since $T_{l_{jk}} \cdot \nu = -\nu \cdot T_{l_{jk}}$, $T_{l_{jk}} \cdot \omega = -\omega \cdot T_{l_{jk}}$ and $\omega \cdot \nu = \nu \cdot \omega$. Switching the indices j and k we also get

$$\begin{aligned} T_k \cdot T_j + (f_j T_k - f_k T_j) \cdot \nu - f_k f_j &= \epsilon_{kj} (f_{l_{kj}} - T_{l_{kj}} \cdot \nu) \cdot \omega \\ &= -\epsilon_{jk} (f_{l_{jk}} - T_{l_{jk}} \cdot \nu) \cdot \omega \end{aligned}$$

since $\epsilon_{kj} = -\epsilon_{jk}$ and $l_{kj} = l_{jk}$. We deduce that

$$\frac{1}{2} (T_j \cdot T_k - T_k \cdot T_j) + (f_k T_j - f_j T_k) \cdot \nu = \epsilon_{jk} (f_{l_{jk}} - T_{l_{jk}} \cdot \nu) \cdot \omega.$$

The result is then a consequence of Lemma 5.1 together with the relation

$$\left(\tilde{\Gamma}(X) \cdot \psi \right)^* = \Gamma(X) \cdot \psi^*$$

and the first property in (50). \square

5.4.1. *The metric Lie group S^3 .* A spinor representation of a surface immersed in S^3 was already given in [16] (see also [3, 4]). We give here a spinor representation relying on the group structure; it appears that it coincides with the result in [16].

We regard the sphere S^3 as the set of the unit quaternions, with its natural group structure. The Lie algebra of S^3 identifies to \mathbb{R}^3 , with the bracket $[X, Y] = 2X \times Y$ for all $X, Y \in \mathbb{R}^3$ (\times is the usual cross product). By the Koszul formula (4), for all $X, Y \in \mathbb{R}^3$,

$$\Gamma(X)(Y) = X \times Y.$$

As a bivector, for all $X = X_1 e_1^o + X_2 e_2^o + X_3 e_3^o \in \mathbb{R}^3$,

$$\begin{aligned} \Gamma(X) &= \frac{1}{2} (e_1^o \cdot \Gamma(X)(e_1^o) + e_2^o \cdot \Gamma(X)(e_2^o) + e_3^o \cdot \Gamma(X)(e_3^o)) \\ &= X_1 e_2^o \cdot e_3^o + X_2 e_3^o \cdot e_1^o + X_3 e_1^o \cdot e_2^o \\ &= -X \cdot (e_1^o \cdot e_2^o \cdot e_3^o). \end{aligned}$$

Thus, if $\varphi \in \tilde{Q} \times_\rho Cl_3^0$ represents an immersion of an oriented surface M in S^3 and if $\psi \in \Gamma(\Sigma M)$ is such that $\varphi = \psi^*$, then, for all $X \in TM$,

$$\begin{aligned} \Gamma(X) \cdot \varphi &= -X \cdot (e_1^o \cdot e_2^o \cdot e_3^o) \cdot \varphi \\ &= -X \cdot \omega \cdot \nu \cdot \varphi \\ &= (X \cdot \nu) \cdot \omega \cdot \varphi \\ &= (X \cdot \omega \cdot \psi)^* \end{aligned}$$

where ω is the area form of M , and ν is the vector normal to M in S^3 . Since $\varphi \in \Gamma(U\Sigma)$ is a solution of (24), $\psi \in \Gamma(\Sigma M)$ is a solution of

$$\nabla_X \psi = -\frac{1}{2} S(X) \cdot \psi + \frac{1}{2} X \cdot \omega \cdot \psi$$

and satisfies $|\psi| = 1$. Taking the trace, we get

$$\begin{aligned} D\psi &= e_1 \cdot \nabla_{e_1} \psi + e_2 \cdot \nabla_{e_2} \psi \\ &= H\psi - \omega \cdot \psi \end{aligned}$$

where (e_1, e_2) is a positively oriented and orthonormal basis of TM . Now, setting $\overline{\psi} = \psi^+ - \psi^-$ and since $\omega \cdot \psi = -i\overline{\psi}$ (recall that $i\omega$ acts as the identity on $\Sigma^+ M$ and as -identity on $\Sigma^- M$), we get

$$D\psi = H\psi - i\overline{\psi},$$

which is also the spinor characterization given by Morel in [16].

5.4.2. *Surfaces in the 3-dimensional metric Lie groups $E(\kappa, \tau)$, $\tau \neq 0$.* We recover here a spinor characterization of immersions in the 3-dimensional homogeneous spaces $E(\kappa, \tau)$; this result was obtained by the second author in [19], using a characterization of immersions in these spaces by Daniel [7]. We give here an independent proof, and rather obtain the result of Daniel as a corollary.

The metric Lie group $E(\kappa, \tau)$, $\tau \neq 0$, is defined as follows: its Lie algebra is $\mathcal{G} = \mathbb{R}^3$, with the bracket defined on the vectors e_1^o, e_2^o, e_3^o of the canonical basis by

$$[e_1^o, e_2^o] = 2\tau e_3^o, \quad [e_2^o, e_3^o] = \sigma e_1^o, \quad [e_3^o, e_1^o] = \sigma e_2^o$$

where $\sigma = \frac{\kappa}{2\tau}$. The metric on \mathcal{G} is the canonical metric, ie the metric such that the basis (e_1^o, e_2^o, e_3^o) is orthonormal. The Levi-Civita connection is then given by

$$(55) \quad \Gamma(X)(Y) = \{\tau(X - \langle X, e_3^o \rangle e_3^o) + (\sigma - \tau)\langle X, e_3^o \rangle e_3^o\} \times Y$$

for $X, Y \in \mathcal{G}$; see e.g. [7].

Let $S : TM \rightarrow TM$ be a symmetric operator. We assume that a vector field $T \in \Gamma(TM)$ and a function $f \in C^\infty(M, \mathbb{R})$ are given such that

$$(56) \quad |T|^2 + f^2 = 1,$$

$$(57) \quad \nabla_X T = f(S(X) - \tau JX)$$

and

$$(58) \quad df(X) = -\langle S(X) - \tau JX, T \rangle$$

for all $X \in TM$, where $J : TM \rightarrow TM$ denotes the rotation of angle $+\pi/2$ in the tangent planes.

Theorem 5. [19] *If M is simply connected, the following two statements are equivalent:*

(1) *There exists $\psi \in \Gamma(\Sigma M)$ such that $|\psi| = 1$ and*

$$(59) \quad \nabla_X \psi = -\frac{1}{2}S(X) \cdot \psi + \frac{1}{2}\{(2\tau - \sigma)\langle X, T \rangle (T - f) - \tau X\} \cdot \omega \cdot \psi$$

for all $X \in TM$.

(2) *There exists an isometric immersion of M into $E(\kappa, \tau)$, with shape operator S .*

Proof. We consider the trivial line bundle $E = \mathbb{R}\nu$, where ν is a unit section. The bundle $TM \oplus E$ is of rank 3, and is assumed to be oriented by the orientation of TM and by ν . We suppose that it is endowed with the natural product metric. Let us denote by \times the natural cross product in the fibers. We set

$$\underline{e}_3 = T + f\nu,$$

and, for all $X, Y \in TM \oplus E$,

$$(60) \quad \Gamma(X)(Y) = \{\tau(X - \langle X, \underline{e}_3 \rangle \underline{e}_3) + (\sigma - \tau)\langle X, \underline{e}_3 \rangle \underline{e}_3\} \times Y.$$

Defining $B : TM \times TM \rightarrow E$ and its adjoint $B^* : TM \times E \rightarrow TM$ by

$$(61) \quad B(X, Y) = \langle S(X), Y \rangle \nu \quad \text{and} \quad B^*(X, \nu) = S(X)$$

for all $X, Y \in TM$, the equations (57) and (58) are equivalent to the single equation

$$(62) \quad \nabla_X \underline{e}_3 = \Gamma(X)(\underline{e}_3) - B(X, \underline{e}_3^T) + B^*(X, \underline{e}_3^N)$$

for all $X \in TM$, where ∇ is the sum of the Levi-Civita connection on TM and the trivial connection on E . This is (20) for $Z = \underline{e}_3$. We will need the following expression for Γ :

Lemma 5.3. *For all $X \in TM$, the linear map $\Gamma(X) : TM \oplus E \rightarrow TM \oplus E$ defined by (60) is represented by the bivector*

$$\Gamma(X) = \{(2\tau - \sigma)\langle X, T \rangle (T \cdot \nu - f) - \tau X \cdot \nu\} \cdot \omega.$$

Proof. The linear map $\Gamma(X)$ is represented by the bivector

$$\Gamma(X) = \frac{1}{2} (\underline{e}_1 \cdot \Gamma(X)(\underline{e}_1) + \underline{e}_2 \cdot \Gamma(X)(\underline{e}_2) + \underline{e}_3 \cdot \Gamma(X)(\underline{e}_3))$$

where $\underline{e}_1, \underline{e}_2$ are such that $\underline{e}_1, \underline{e}_2, \underline{e}_3$ is a positively oriented and orthonormal basis of $TM \oplus E$ (see Lemma A.1); thus, a straightforward computation shows that $\Gamma(X)$ is represented by the bivector

$$(63) \quad \Gamma(X) = -\tau(X \times \underline{e}_3) \cdot \underline{e}_3 + (\sigma - \tau) \langle X, \underline{e}_3 \rangle \underline{e}_1 \cdot \underline{e}_2.$$

The following formula may be checked by a direct computation: for $X, Y \in TM \oplus E$,

$$X \times Y = -(X \cdot Y + \langle X, Y \rangle) \underline{e}_1 \cdot \underline{e}_2 \cdot \underline{e}_3;$$

this gives

$$\begin{aligned} (X \times \underline{e}_3) \cdot \underline{e}_3 &= -(X \cdot \underline{e}_3 + \langle X, \underline{e}_3 \rangle) \underline{e}_1 \cdot \underline{e}_2 \cdot \underline{e}_3 \cdot \underline{e}_3 \\ &= (X - \langle X, \underline{e}_3 \rangle \underline{e}_3) \underline{e}_1 \cdot \underline{e}_2 \cdot \underline{e}_3 \\ &= (X - \langle X, T \rangle (T + f\nu)) \cdot \omega \cdot \nu \\ &= (X \cdot \nu - \langle X, T \rangle (T \cdot \nu - f)) \cdot \omega. \end{aligned}$$

Moreover,

$$\begin{aligned} \langle X, \underline{e}_3 \rangle \underline{e}_1 \cdot \underline{e}_2 &= \langle X, T \rangle (-\underline{e}_1 \cdot \underline{e}_2 \cdot \underline{e}_3 \cdot \underline{e}_3) \\ &= \langle X, T \rangle (-\omega \cdot \nu \cdot (T + f\nu)) \\ &= -\langle X, T \rangle (T \cdot \nu - f) \cdot \omega. \end{aligned}$$

Plugging these two formulas in (63) we get the result. \square

We deduce the following key lemma:

Lemma 5.4. *A spinor field $\varphi \in \Gamma(U\Sigma)$ solution of (24) is equivalent to a spinor field $\psi \in \Gamma(\Sigma M)$ solution of (59).*

Proof. We use the identification $\psi \in \Gamma(\Sigma M) \mapsto \psi^* \in \Gamma(\Sigma)$ described at the beginning of the section; we recall that, for all $X \in TM$,

$$(64) \quad (\nabla_X \psi)^* = \nabla_X(\psi^*) \quad \text{and} \quad (X \cdot \psi)^* = X \cdot \nu \cdot (\psi^*).$$

Thus, if $\varphi \in \Gamma(U\Sigma)$ is a solution of (24) and if $\psi \in \Gamma(\Sigma M)$ is such that $\psi^* = \varphi$, using (64) together with the formula

$$\sum_{j=1}^p e_j \cdot B(X, e_j) = \sum_{j=1}^p e_j \cdot \langle S(X), e_j \rangle \nu = S(X) \cdot \nu$$

and Lemma 5.3, we get:

$$\begin{aligned} (\nabla_X \psi)^* &= \nabla_X \varphi \\ &= -\frac{1}{2} S(X) \cdot \nu \cdot \varphi + \frac{1}{2} \{ (2\tau - \sigma) \langle X, T \rangle (T \cdot \nu - f) - \tau X \cdot \nu \} \cdot \omega \cdot \varphi \\ &= \left(-\frac{1}{2} S(X) \cdot \psi + \frac{1}{2} \{ (2\tau - \sigma) \langle X, T \rangle (T - f) - \tau X \} \cdot \omega \cdot \psi \right)^*. \end{aligned}$$

This gives (59). Reciprocally, if ψ is a solution of (59), the spinor field $\varphi = \psi^*$ satisfies (24). This proves the lemma. \square

Instead of $\psi \in \Gamma(\Sigma M)$ solution of (59) we may thus consider $\varphi \in \Gamma(U\Sigma)$ solution of (24). Theorem 5 will thus be a consequence of Theorem 1 if we can define a bundle isomorphism $f : TM \oplus E \rightarrow M \times \mathcal{G}$ such that (19) and (20) hold. Let us set

$$f(Z) = \langle \langle Z \cdot \varphi, \varphi \rangle \rangle.$$

We first observe that $f(\underline{e}_3)$ is constant: indeed, for all $X \in TM$,

$$\partial_X(f(\underline{e}_3)) = \langle \langle \nabla_X \underline{e}_3 \cdot \varphi, \varphi \rangle \rangle + \langle \langle \underline{e}_3 \cdot \nabla_X \varphi, \varphi \rangle \rangle + \langle \langle \underline{e}_3 \cdot \varphi, \nabla_X \varphi \rangle \rangle = 0$$

in view of (62), (24) and the computation in (29). Moreover, since f preserves the norm of the vectors, $f(\underline{e}_3)$ is a unit vector. Replacing φ by $\varphi \cdot a$ for some $a \in \text{Spin}(\mathcal{G})$ if necessary, we may thus assume that $f(\underline{e}_3) = e_3^o$. We now check (19): since the map f is an orientation preserving isometry and using $f(\underline{e}_3) = e_3^o$, we have, for all $X, Y \in TM$,

$$\begin{aligned} f(\Gamma(X)(Y)) &= f(\{\tau(X - \langle X, \underline{e}_3 \rangle \underline{e}_3) + (\sigma - \tau)\langle X, \underline{e}_3 \rangle \underline{e}_3\} \times Y) \\ &= \{\tau(f(X) - \langle f(X), f(\underline{e}_3) \rangle f(\underline{e}_3)) + (\sigma - \tau)\langle f(X), f(\underline{e}_3) \rangle f(\underline{e}_3)\} \times f(Y) \\ &= \{\tau(f(X) - \langle f(X), e_3^o \rangle e_3^o) + (\sigma - \tau)\langle f(X), e_3^o \rangle e_3^o\} \times f(Y) \\ &= \Gamma(f(X))(f(Y)). \end{aligned}$$

Finally, the proof of (20) is very similar to the proof of this identity made in Section 5.1 for $G = \mathbb{R}^n$: we only have to add the term involving Γ which appears in the expression (24) of the covariant derivative of φ ; we leave the details to the reader. \square

Remark 8. We also get an explicit representation formula: the immersion is given by the Darboux integral of $\xi : X \mapsto \langle \langle X \cdot \varphi, \varphi \rangle \rangle$, which may be written in terms of ψ by the formula (54).

We deduce the following result, first obtained by Daniel in [7] using the moving frame method:

Corollary 2. If S , T , f , κ and τ satisfy (56)-(58), the Gauss equation

$$(65) \quad K = \det S + \tau^2 + (\kappa - 4\tau^2) f^2$$

and the Codazzi equation

$$(66) \quad \nabla_X(SY) - \nabla_Y(SX) - S([X, Y]) = (\kappa - 4\tau^2)f(\langle Y, T \rangle X - \langle X, T \rangle Y),$$

then there exists an isometric immersion of M into $E(\kappa, \tau)$ with shape operator S . Moreover the immersion is unique up to a global isometry of $E(\kappa, \tau)$ preserving the orientations.

Proof. The equations (65) and (66) are equivalent to the Gauss and Codazzi equations (33) and (35) where B is defined by (61). They are thus exactly the integrability conditions for (24), and consequently also for (59). \square

5.4.3. Three-dimensional semi-direct products. We consider here a semi-direct product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

if (e_1^o, e_2^o, e_3^o) stands for the canonical basis of $\mathcal{G} = \mathbb{R}^2 \times \mathbb{R}$, the Lie bracket is given by

$$[e_1^o, e_2^o] = 0, \quad [e_3^o, e_1^o] = ae_1^o + ce_2^o, \quad [e_3^o, e_2^o] = be_1^o + de_2^o.$$

We equip $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with the left invariant metric such that (e_1^o, e_2^o, e_3^o) is orthonormal. By the Koszul formula, we get

$$(67) \quad \nabla_{e_1^o} e_1^o = a e_3^o, \quad \nabla_{e_1^o} e_2^o = \frac{b+c}{2} e_3^o, \quad \nabla_{e_1^o} e_3^o = -a e_1^o - \frac{b+c}{2} e_2^o,$$

$$(68) \quad \nabla_{e_2^o} e_1^o = \frac{b+c}{2} e_3^o, \quad \nabla_{e_2^o} e_2^o = d e_3^o, \quad \nabla_{e_2^o} e_3^o = -\frac{b+c}{2} e_1^o - d e_2^o$$

and

$$(69) \quad \nabla_{e_3^o} e_1^o = \frac{c-b}{2} e_2^o, \quad \nabla_{e_3^o} e_2^o = \frac{b-c}{2} e_1^o, \quad \nabla_{e_3^o} e_3^o = 0,$$

and deduce

$$\Gamma(X) = \left(aX_1 + \frac{b+c}{2} X_2 \right) e_1^o \cdot e_3^o + \left(\frac{b+c}{2} X_1 + dX_2 \right) e_2^o \cdot e_3^o + \frac{c-b}{2} X_3 e_1^o \cdot e_2^o$$

for all $X \in \mathcal{G}$. We first assume that M is an oriented surface in $G = \mathbb{R}^2 \rtimes_A \mathbb{R}$. Recalling that

$$(T_j + f_j \nu) \cdot (T_k + f_k \nu) = \epsilon_{jk} (f_{l_{jk}} - T_{l_{jk}} \cdot \nu) \cdot \omega$$

(see the proof of Lemma 5.2), we obtain

$$\begin{aligned} \Gamma(X) &= - \left(aX_1 + \frac{b+c}{2} X_2 \right) (f_2 - T_2 \cdot \nu) \cdot \omega \\ &\quad + \left(\frac{b+c}{2} X_1 + dX_2 \right) (f_1 - T_1 \cdot \nu) \cdot \omega + \frac{c-b}{2} X_3 (f_3 - T_3 \cdot \nu) \cdot \omega \end{aligned}$$

and

$$(70) \quad \tilde{\Gamma}(X) = - \left(aX_1 + \frac{b+c}{2} X_2 \right) (f_2 - T_2) \cdot \omega \\ + \left(\frac{b+c}{2} X_1 + dX_2 \right) (f_1 - T_1) \cdot \omega + \frac{c-b}{2} X_3 (f_3 - T_3) \cdot \omega.$$

Conversely, we consider an oriented Riemannian surface M , and a symmetric operator $S : TM \rightarrow TM$. We suppose that there exist tangent vectors fields $T_i \in \Gamma(TM)$ and functions $f_i \in C^\infty(M)$ for $1 \leq i \leq 3$ satisfying

$$(71) \quad \langle T_i, T_j \rangle + f_i f_j = \delta_i^j$$

for all $1 \leq i, j \leq 3$, and the equations (22) and (23) in Remark 2, with the coefficients Γ_{ij}^k given by (67)-(69). Theorem 4 then yields the following result:

Theorem 6. *If M is simply connected, the following two statements are equivalent:*

- (1) *there exists an isometric immersion of M into $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with shape operator S ;*
- (2) *there exists $\psi \in \Gamma(\Sigma M)$ such that $|\psi| = 1$ and*

$$(72) \quad \nabla_X \psi = -\frac{1}{2} S(X) \cdot \psi + \frac{1}{2} \tilde{\Gamma}(X) \cdot \psi$$

for all $X \in TM$.

The metric Lie group Sol_3 . Now, we describe the special case of a surface in Sol_3 : this achieves the spinor representation of immersions of surfaces into 3-dimensional Riemannian homogeneous spaces [19].

Let us recall that Sol_3 is the only metric Lie group whose isometry group is 3-dimensional. It is defined as follows: its Lie algebra is $\mathcal{G} = \mathbb{R}^3$, with the bracket defined on the canonical basis (e_1^o, e_2^o, e_3^o) by

$$[e_1^o, e_2^o] = 0, \quad [e_2^o, e_3^o] = -e_2^o, \quad [e_3^o, e_1^o] = -e_1^o.$$

This is the semi-direct product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with $a = -1, b = c = 0, d = 1$. The metric on \mathcal{G} is the canonical metric, i.e., the metric such that the basis (e_1^o, e_2^o, e_3^o) is orthonormal. By the formulas (67)-(69), the Levi-Civita connection is then such that

$$(73) \quad \Gamma_{11}^3 = -\Gamma_{13}^1 = -1, \quad \Gamma_{22}^3 = -\Gamma_{23}^2 = 1$$

and $\Gamma_{ij}^k = 0$ for the other indices.

Let us consider an oriented Riemannian surface M , and a symmetric operator $S : TM \rightarrow TM$. We suppose that there exist tangent vectors fields $T_i \in \Gamma(TM)$ and functions $f_i \in C^\infty(M)$ for $1 \leq i \leq 3$ satisfying

$$(74) \quad \langle T_i, T_j \rangle + f_i f_j = \delta_i^j$$

for all $1 \leq i, j \leq 3$, and, for all $X \in TM$,

$$(75) \quad \begin{aligned} \nabla_X T_i &= (-1)^i \langle X, T_i \rangle T_3 + f_i S(X), \\ df_i(X) &= (-1)^i \langle X, T_i \rangle f_3 - \langle SX, T_i \rangle \end{aligned}$$

for $1 \leq i \leq 2$,

$$(76) \quad \begin{aligned} \nabla_X T_3 &= \sum_{j=1}^2 (-1)^{j+1} \langle X, T_j \rangle T_j + f_3 S(X), \\ df_3(X) &= \sum_{j=1}^2 (-1)^{j+1} \langle X, T_j \rangle f_j - \langle S(X), T_3 \rangle. \end{aligned}$$

The equations (75) and (76) are the equations (22) and (23) in Remark 2, with the coefficients Γ_{ij}^k given by (73). According to (70) with $a = -1, b = c = 0$ and $d = 1$ we set

$$(77) \quad \tilde{\Gamma}(X) = \{ \langle X, T_1 \rangle (f_2 - T_2) + \langle X, T_2 \rangle (f_1 - T_1) \} \cdot \omega$$

for all $X \in TM$. Theorem 6 then gives a spinor characterization of an immersion in Sol_3 .

As a corollary, we obtain a new proof of a result by Lodovici [11] concerning existence and uniqueness of isometric immersions in Sol_3 , since equation (72) is solvable if and only if the equations of Gauss and Codazzi hold (see Section 4).

$\mathbb{H}^2 \times \mathbb{R}$ as a metric Lie group. Finally, viewing $\mathbb{H}^2 \times \mathbb{R}$ as a metric Lie group, we obtain a new spinor characterization of an immersion in $\mathbb{H}^2 \times \mathbb{R}$ which differs from [19] where the product point of view was used.

We recall that $\mathbb{H}^2 \times \mathbb{R}$ is the semi-direct product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with $a = 1, b = c = d = 0$.

The metric on \mathcal{G} is the canonical metric, i.e., the metric such that the basis (e_1^o, e_2^o, e_3^o) is orthonormal. Lie bracket is given by

$$[e_1^o, e_2^o] = 0, \quad [e_3^o, e_1^o] = e_1^o, \quad [e_3^o, e_2^o] = 0.$$

By the formulas (67)-(69), the Levi-Civita connection is then such that

$$(78) \quad \Gamma_{11}^3 = -\Gamma_{13}^1 = 1$$

and $\Gamma_{ij}^k = 0$ for the other indices.

Let us consider an oriented Riemannian surface M , and a symmetric operator $S : TM \rightarrow TM$. We suppose that there exist tangent vectors fields $T_i \in \Gamma(TM)$ and functions $f_i \in C^\infty(M)$ for $1 \leq i \leq 3$ satisfying

$$(79) \quad \langle T_i, T_j \rangle + f_i f_j = \delta_i^j$$

for all $1 \leq i, j \leq 3$, and, for all $X \in TM$,

$$(80) \quad \begin{aligned} \nabla_X T_1 &= \langle X, T_1 \rangle T_3 + f_1 S(X), \\ df_1(X) &= \langle X, T_1 \rangle f_3 - \langle SX, T_1 \rangle, \end{aligned}$$

$$(81) \quad \begin{aligned} \nabla_X T_2 &= f_2 S(X), \\ df_2(X) &= -\langle SX, T_2 \rangle, \end{aligned}$$

$$(82) \quad \begin{aligned} \nabla_X T_3 &= -\langle X, T_3 \rangle T_1 + f_3 S(X), \\ df_3(X) &= -\langle X, T_3 \rangle f_1 - \langle SX, T_3 \rangle. \end{aligned}$$

With these identities and according to (70) with $a = 1, b = c = d = 0$, we set

$$(83) \quad \tilde{\Gamma}(X) = -\langle X, T_1 \rangle (f_2 - T_2) \cdot \omega$$

for all $X \in TM$. Theorem 6 then gives a spinor characterization of an immersion in $\mathbb{H}^2 \times \mathbb{R}$.

5.5. CMC-surfaces in a 3-dimensional metric Lie group. The aim here is to show that the representation formula for CMC-surfaces in a 3-dimensional metric Lie group by Meeks, Mira, Perez and Ros [13, Theorem 3.12] may be obtained as a consequence of the general representation formula in Theorem 1. For sake of brevity we assume that the group G is unimodular and only give the principal arguments, without details. Under this hypothesis, there exists an orthonormal basis (e_1^o, e_2^o, e_3^o) of the Lie algebra \mathcal{G} and constants $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ such that the Levi-Civita connection of G is given by

$$\begin{aligned} \Gamma(X)(e_1^o) &:= \nabla_X e_1^o = X_3 \mu_3 e_2^o - X_2 \mu_2 e_3^o, \\ \Gamma(X)(e_2^o) &:= \nabla_X e_2^o = -X_3 \mu_3 e_1^o + X_1 \mu_1 e_3^o, \\ \Gamma(X)(e_3^o) &:= \nabla_X e_3^o = X_2 \mu_2 e_1^o - X_1 \mu_1 e_2^o \end{aligned}$$

(see e.g. [13, Section 2.6]), i.e.

$$(84) \quad \begin{aligned} \Gamma(X) &= \frac{1}{2} (e_1^o \cdot \Gamma(X)(e_1^o) + e_2^o \cdot \Gamma(X)(e_2^o) + e_3^o \cdot \Gamma(X)(e_3^o)) \\ &= X_1 \mu_1 e_2^o \cdot e_3^o + X_2 \mu_2 e_3^o \cdot e_1^o + X_3 \mu_3 e_1^o \cdot e_2^o \end{aligned}$$

for all $X \in \mathcal{G}$. Following [13] we introduce the H -potential of the group G

$$(85) \quad R(g) = H(1 + |g|^2)^2 - \frac{i}{2} (\mu_1 |1 - g^2|^2 + \mu_2 |1 + g^2|^2 + 4\mu_3 |g|^4)$$

for all $g \in \overline{\mathbb{C}}$. The importance of this quantity appears in the following lemma, which will permit to express the right-hand side of the Dirac equation (27):

Lemma 5.5. *Let us consider a positively oriented and orthonormal basis e_1, e_2, ν of \mathcal{G} and set, for $\nu = \nu_1 e_1^o + \nu_2 e_2^o + \nu_3 e_3^o$,*

$$(86) \quad T(\nu) = \mu_1 \nu_1 e_1^o + \mu_2 \nu_2 e_2^o + \mu_3 \nu_3 e_3^o,$$

$A = \frac{1}{2} \langle e_2, T(\nu) \rangle$ and $B = -\frac{1}{2} \langle e_1, T(\nu) \rangle$. Then, if

$$g = \frac{\nu_1 + i\nu_2}{1 + \nu_3}$$

is the stereographic projection of $\nu \in S^2$ with respect to the south pole $-e_3^o$ of S^2 , we have

$$H\nu + \frac{1}{2} (e_1 \cdot \Gamma(e_1) + e_2 \cdot \Gamma(e_2)) = \frac{1}{(1 + |g|^2)^2} (\Re R(g) - \Im R(g) e_1 \cdot e_2) \cdot \nu + Ae_1 + Be_2.$$

Proof. For $i \in \{1, 2, 3\}$, let us denote by

$$p(e_i^o) := \langle e_i^o, e_1 \rangle e_1 + \langle e_i^o, e_2 \rangle e_2$$

the orthogonal projection of the vector e_i^o onto the plane generated by e_1 and e_2 . By (84) we have

$$e_1 \cdot \Gamma(e_1) + e_2 \cdot \Gamma(e_2) = \mu_1 p(e_1^o) \cdot e_2^o \cdot e_3^o + \mu_2 p(e_2^o) \cdot e_3^o \cdot e_1^o + \mu_3 p(e_3^o) \cdot e_1^o \cdot e_2^o.$$

The proof is then a direct and long computation using that $p(e_i^o) = e_i^o - \langle e_i^o, \nu \rangle \nu$ together with the formulas

$$(87) \quad \nu_1 = \frac{2 \Re g}{1 + |g|^2}, \quad \nu_2 = \frac{2 \Im g}{1 + |g|^2}, \quad \nu_3 = \frac{1 - |g|^2}{1 + |g|^2}.$$

□

We consider the Clifford map

$$(88) \quad \begin{aligned} \mathcal{G} &\rightarrow \mathbb{H}(2) \\ x_1 e_1^o + x_2 e_2^o + x_3 e_3^o &\mapsto \begin{pmatrix} ix_3 + j(x_1 - ix_2) & 0 \\ 0 & -ix_3 - j(x_1 - ix_2) \end{pmatrix} \end{aligned}$$

which identifies \mathcal{G} to the imaginary quaternions so that

$$(89) \quad e_1^o \simeq \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix} \simeq j, \quad e_2^o \simeq \begin{pmatrix} -ji & 0 \\ 0 & ji \end{pmatrix} \simeq -ji, \quad e_3^o \simeq \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \simeq i.$$

It identifies $Cl(\mathcal{G})$ to the set of matrices

$$(90) \quad \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad a, b \in \mathbb{H} \right\}$$

and $Spin(\mathcal{G})$ to the group of unit quaternions

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad a \in \mathbb{H}, |a| = 1 \right\} \simeq \{a \in \mathbb{H}, |a| = 1\}.$$

We choose a conformal parameter $z = x + iy$ of the surface, and denote by μ the real function such that the metric is $\mu^2(dx^2 + dy^2)$. In a spinorial frame above the orthonormal frame $e_1 = \frac{1}{\mu} \partial_x$, $e_2 = \frac{1}{\mu} \partial_y$, the spinor field φ is represented by $[\varphi] = z_1 + jz_2$ where $z_1, z_2 \in \mathbb{C}$ are such that $|z_1|^2 + |z_2|^2 = 1$.

Lemma 5.6. *The Dirac equation (27) is equivalent to the system*

$$(91) \quad \frac{1}{\sqrt{\mu}} \partial_{\bar{z}} (\sqrt{\mu} \bar{z}_1) = i \frac{\mu}{2} \frac{\overline{R(g)}}{(1+|g|^2)^2} \bar{z}_2 + \frac{\mu}{2} (A + iB) \bar{z}_1$$

$$(92) \quad \frac{1}{\sqrt{\mu}} \partial_{\bar{z}} (\sqrt{\mu} z_2) = -i \frac{\mu}{2} \frac{\overline{R(g)}}{(1+|g|^2)^2} z_1 + \frac{\mu}{2} (A + iB) z_2.$$

Moreover, the \mathcal{G} -valued 1-form ξ in Theorem 1 is

$$(93) \quad \begin{aligned} \xi(X) = & i \{ 2x_1 \Im m(z_1 \bar{z}_2) - 2x_2 \Re e(z_1 \bar{z}_2) + x_3 (|z_1|^2 - |z_2|^2) \} \\ & + j \{ x_1(z_1^2 + z_2^2) - ix_2(z_1^2 - z_2^2) - 2ix_3 z_1 z_2 \} \end{aligned}$$

for all $X = x_1 e_1 + x_2 e_2 + x_3 \nu \in TM \oplus E$.

Proof. We use here the identification $\psi \in \Gamma(\Sigma M) \mapsto \psi^* \in \Gamma(\Sigma)$ satisfying the properties (64): according to Lemma 5.5, the spinor field $\psi \in \Gamma(\Sigma M)$ such that $\psi^* = \varphi$ is solution of

$$(94) \quad D\psi = \frac{1}{(1+|g|^2)^2} (\Re e R(g) - \Im m R(g) e_1 \cdot e_2) \cdot \psi + (Ae_1 + Be_2) \cdot \psi.$$

We identify Cl_2 to \mathbb{H} using the Clifford map

$$(95) \quad \begin{aligned} \mathbb{R}^2 & \rightarrow \mathbb{H} \\ (x_1, x_2) & \mapsto j(x_1 - ix_2) \end{aligned}$$

so that, in the fixed spinorial frame above $e_1 = \frac{1}{\mu} \partial_x$, $e_2 = \frac{1}{\mu} \partial_y$,

$$[e_1] = j, \quad [e_2] = -ji, \quad [e_1 \cdot e_2] = i.$$

Using moreover that

$$[\nabla_{\partial_x} \psi] = \partial_x [\psi] - \frac{i}{2\mu} \partial_y \mu [\psi] \quad [\nabla_{\partial_y} \psi] = \partial_y [\psi] + \frac{i}{2\mu} \partial_x \mu [\psi]$$

(by (13), and the computation of the Christoffel symbols), the left-hand side of (94) is

$$[D\psi] = \frac{1}{\mu} j \left\{ \partial_x [\psi] - \frac{i}{2\mu} \partial_y \mu [\psi] \right\} - \frac{1}{\mu} ji \left\{ \partial_y [\psi] + \frac{i}{2\mu} \partial_x \mu [\psi] \right\}$$

whereas the right-hand side is

$$\left(\frac{\overline{R(g)}}{(1+|g|^2)^2} + j(A - iB) \right) [\psi].$$

We finally need to precise the identification $\psi \mapsto \psi^*$: in spinorial frames above e_1, e_2 and e_1, e_2, ν , since the second property in (64) is required and using the Clifford maps (88) and (95), it is not difficult to see that the map $\psi \mapsto \psi^*$ corresponds to the map

$$u + jv \mapsto \begin{pmatrix} u + jiv & 0 \\ 0 & u + jiv \end{pmatrix};$$

ψ is thus represented by the quaternion $[\psi] = z_1 - jiz_2$. Direct computations then give the system (91)-(92).

Expression (93) also follows from a direct computation: we have, in Cl_3 ,

$$\begin{aligned} \xi(X) &= \tau[\varphi][X][\varphi] \\ &\simeq (\bar{z}_1 - j\bar{z}_2)(ix_3 + j(x_1 - ix_2))(z_1 + jz_2), \end{aligned}$$

which easily gives the result. \square

We set

$$(96) \quad g = i \frac{\bar{z}_2}{z_1}, \quad f = -2\mu z_1^2.$$

The function g is the left invariant Gauss map of the surface, stereografically projected with respect to the south pole of S^2 , since

$$\nu = i(|z_1|^2 - |z_2|^2) - 2jiz_1z_2$$

is a unit vector normal to the immersion ($x_1 = x_2 = 0$ and $x_3 = 1$ in (93)) and

$$\frac{\nu_1 + i\nu_2}{1 + \nu_3} = \frac{2i \bar{z}_1 \bar{z}_2}{1 + |z_1|^2 - |z_2|^2} = \frac{2i \bar{z}_1 \bar{z}_2}{2|z_1|^2} = i \frac{\bar{z}_2}{z_1}.$$

Direct computations then show that equations (91)-(92) are equivalent to

$$(97) \quad f = 4 \frac{\partial_z g}{R(g)}$$

and

$$(98) \quad \frac{\partial_{\bar{z}} f}{f} = -\frac{2}{1 + |g|^2} \partial_{\bar{z}} \bar{g} g + \mu(A + iB),$$

and that (93) reads

$$(99) \quad \xi = \Re e \left(\frac{1}{2} f(\bar{g}^2 - 1) dz, \frac{i}{2} f(\bar{g}^2 + 1) dz, f \bar{g} dz \right)$$

in (e_1^o, e_2^o, e_3^o) (recall (89)). This last formula is the Weierstrass-type representation given in [13, Theorem 3.15]. Using that

$$A = \left\langle \xi \left(\frac{\partial_y}{\mu} \right), T(\nu) \right\rangle \quad \text{and} \quad B = - \left\langle \xi \left(\frac{\partial_x}{\mu} \right), T(\nu) \right\rangle$$

(Lemma 5.5) together with (99) and (86) we get that

$$(100) \quad A + iB = -\frac{i}{4\mu} \bar{f} (\mu_1 \nu_1 (g^2 - 1) - i\mu_2 \nu_2 (g^2 + 1) + 2\mu_3 \nu_3 g).$$

Differentiating (97) with respect to \bar{z} and using (98) together with (100) and (87) we see by a further computation that g satisfies

$$(101) \quad g_{z\bar{z}} = \frac{R_g}{R} g_z g_{\bar{z}} + \left(\frac{R_{\bar{g}}}{R} - \frac{\overline{R_g}}{R} \right) |g_z|^2,$$

which is the structure equation for the left invariant Gauss map in [13, Theorem 3.15].

APPENDIX A. SKEW-SYMMETRIC OPERATORS AND BIVECTORS

We consider \mathbb{R}^n endowed with its canonical scalar product. A skew-symmetric operator $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ naturally identifies to a bivector $\underline{u} \in \Lambda^2 \mathbb{R}^n$, which may in turn be regarded as belonging to the Clifford algebra $Cl_n(\mathbb{R})$. We precise here the relations between the Clifford product in $Cl_n(\mathbb{R})$ and the composition of endomorphisms. If a and b belong to the Clifford algebra $Cl_n(\mathbb{R})$, we set

$$[a, b] = \frac{1}{2} (a \cdot b - b \cdot a),$$

where the dot \cdot is the Clifford product. We denote by (e_1, \dots, e_n) the canonical basis of \mathbb{R}^n .

Lemma A.1. *Let $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a skew-symmetric operator. Then the bivector*

$$(102) \quad \underline{u} = \frac{1}{2} \sum_{j=1}^n e_j \cdot u(e_j) \in \Lambda^2 \mathbb{R}^n \subset Cl_n(\mathbb{R})$$

represents u , and, for all $\xi \in \mathbb{R}^n$,

$$[\underline{u}, \xi] = u(\xi).$$

In the paper, and for sake of simplicity, we will use the same letter u to denote \underline{u} .

Proof. For $i < j$, we consider the linear map

$$u : \quad e_i \mapsto e_j, \quad e_j \mapsto -e_i, \quad e_k \mapsto 0 \quad \text{if } k \neq i, j;$$

it is skew-symmetric and corresponds to the bivector $e_i \wedge e_j \in \Lambda^2 \mathbb{R}^n$; it is thus naturally represented by $\underline{u} = e_i \cdot e_j = \frac{1}{2} (e_i \cdot e_j - e_j \cdot e_i)$, which is (102). We then compute, for $k = 1, \dots, n$,

$$[\underline{u}, e_k] = \frac{1}{2} (e_i \cdot e_j \cdot e_k - e_k \cdot e_i \cdot e_j)$$

and easily get

$$[\underline{u}, e_k] = e_j \quad \text{if } k = i, \quad -e_i \quad \text{if } k = j, \quad 0 \quad \text{if } k \neq i, j.$$

The result follows by linearity. \square

Lemma A.2. *Let $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $v : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be two skew-symmetric operators, represented in $Cl_n(\mathbb{R})$ by*

$$u = \frac{1}{2} \sum_{j=1}^n e_j \cdot u(e_j) \quad \text{and} \quad v = \frac{1}{2} \sum_{j=1}^n e_j \cdot v(e_j)$$

respectively. Then $[u, v] \in \Lambda^2 \mathbb{R}^n \subset Cl_n(\mathbb{R})$ represents $u \circ v - v \circ u$.

Proof. For $\xi \in \mathbb{R}^n$, the Jacobi equation yields

$$[[u, v], \xi] = [u, [v, \xi]] - [v, [u, \xi]].$$

Thus, using Lemma A.1 repeatedly, $[u, v]$ represents the map

$$\begin{aligned} \xi \mapsto [[u, v], \xi] &= [u, [v, \xi]] - [v, [u, \xi]] \\ &= [u, v(\xi)] - [v, u(\xi)] \\ &= (u \circ v - v \circ u)(\xi), \end{aligned}$$

and the result follows. \square

We now assume that $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$, $p + q = n$.

Lemma A.3. *Let us consider a linear map $u : \mathbb{R}^p \rightarrow \mathbb{R}^q$ and its adjoint $u^* : \mathbb{R}^q \rightarrow \mathbb{R}^p$. Then the bivector*

$$\underline{u} = \sum_{j=1}^p e_j \cdot u(e_j) \in \Lambda^2 \mathbb{R}^n \subset Cl_n(\mathbb{R})$$

represents

$$\begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix} : \quad \mathbb{R}^p \oplus \mathbb{R}^q \rightarrow \mathbb{R}^p \oplus \mathbb{R}^q,$$

we have

$$(103) \quad \underline{u} = \frac{1}{2} \left(\sum_{j=1}^p e_j \cdot u(e_j) + \sum_{j=p+1}^n e_j \cdot (-u^*(e_j)) \right)$$

and, for all $\xi = \xi_p + \xi_q \in \mathbb{R}^n$,

$$[\underline{u}, \xi] = u(\xi_p) - u^*(\xi_q).$$

As above, we will simply denote \underline{u} by u .

Proof. In view of Lemma A.1, \underline{u} represents the linear map $\xi \mapsto [\underline{u}, \xi]$. We compute, for $\xi \in \mathbb{R}^p$,

$$\begin{aligned} [\underline{u}, \xi] &= \frac{1}{2} \left(\sum_{j=1}^p e_j \cdot u(e_j) \cdot \xi - \xi \cdot \sum_{j=1}^p e_j \cdot u(e_j) \right) \\ &= -\frac{1}{2} \sum_{j=1}^p (e_j \cdot \xi + \xi \cdot e_j) \cdot u(e_j) \\ &= \sum_{j=1}^p \langle \xi, e_j \rangle u(e_j) \\ &= u(\xi), \end{aligned}$$

and, for $\xi \in \mathbb{R}^q$,

$$\begin{aligned} [\underline{u}, \xi] &= \frac{1}{2} \left(\sum_{j=1}^p e_j \cdot u(e_j) \cdot \xi - \xi \cdot \sum_{j=1}^p e_j \cdot u(e_j) \right) \\ &= \frac{1}{2} \sum_{j=1}^p e_j \cdot (u(e_j) \cdot \xi + \xi \cdot u(e_j)) \\ &= -\sum_{j=1}^p e_j \cdot \langle u(e_j), \xi \rangle \\ &= -\sum_{j=1}^p e_j \cdot \langle e_j, u^*(\xi) \rangle \\ &= -u^*(\xi). \end{aligned}$$

Finally,

$$\underline{u} = \sum_{j=1}^p e_j \cdot u(e_j) = \frac{1}{2} \left(\sum_{j=1}^p e_j \cdot u(e_j) + \sum_{j=1}^p -u(e_j) \cdot e_j \right)$$

with

$$\begin{aligned}
\sum_{j=1}^p -u(e_j) \cdot e_j &= - \sum_{i=p+1}^{p+q} \sum_{j=1}^p \langle u(e_j), e_i \rangle e_i \cdot e_j \\
&= \sum_{i=p+1}^{p+q} e_i \cdot \left(- \sum_{j=1}^p \langle e_j, u^*(e_i) \rangle e_j \right) \\
&= \sum_{i=p+1}^{p+q} e_i \cdot (-u^*(e_i)),
\end{aligned}$$

which gives (103). \square

Lemma A.4. *Let us consider two linear maps $u : \mathbb{R}^p \rightarrow \mathbb{R}^q$ and $v : \mathbb{R}^n \rightarrow \mathbb{R}^n$, with v skew-symmetric, and the associated bivectors*

$$u = \sum_{j=1}^p e_j \cdot u(e_j), \quad v = \frac{1}{2} \sum_{j=1}^n e_j \cdot v(e_j).$$

Then $[u, v] \in \Lambda^2 \mathbb{R}^n$ represents the map

$$\xi = \xi_p + \xi_q \mapsto -u^*(v(\xi)_q) + v(u^*(\xi_q)) + u(v(\xi)_p) - v(u(\xi_p)),$$

where the sub-indices p and q mean that we take the components of the vectors in \mathbb{R}^p and \mathbb{R}^q respectively. In view of Lemma A.1, this may also be written in the form

$$[[u, v], \xi] = -u^*(v(\xi)_q) + v(u^*(\xi_q)) + u(v(\xi)_p) - v(u(\xi_p))$$

for all $\xi \in \mathbb{R}^n$.

Proof. From Lemmas A.2 and A.3, the bivector $[u, v] \in \Lambda^2 \mathbb{R}^n$ represents

$$\begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix} \circ v - v \circ \begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix},$$

that is the map

$$\begin{aligned}
\xi &\mapsto \begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix} \begin{pmatrix} v(\xi)_p \\ v(\xi)_q \end{pmatrix} - v \begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix} \begin{pmatrix} \xi_p \\ \xi_q \end{pmatrix} \\
&= \begin{pmatrix} -u^*(v(\xi)_q) + v(u^*(\xi_q)) \\ u(v(\xi)_p) - v(u(\xi_p)) \end{pmatrix},
\end{aligned}$$

which gives the result. \square

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